(c) If any straight line y [illegible] can also be interpreted as then $b = 0$ since $f(0) = 0$. The inequality $|f(x) - mx| > |f(x)|$ can also be interpreted as $|f(x) - mx| > |f(x) - 0|$, i.e. the line $y = 0$ is a better approximation than is $y = mx$.

50. If $m \neq f'(x_0)$ then let $\epsilon = \frac{1}{2}|f'(x_0) - m|$. Then there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $\left|\dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \dfrac{1}{2}|f'(x_0) - m|$. Then $|mx| \leq |f(x) - mx| + |f(x)|$,so $|f(x) - mx| > |mx| - |f(x)| \geq \frac{1}{2}|mx| > |f(x)|$, i.e. $|f(x) - mx| > |f(x)|$, which shows that the straight line $y = 0$ is a better approximating straight line than any other choice $y = mx + b$.

EXERCISE SET 3.3

1. $28x^6$

2. $-36x^{11}$

3. $24x^7 + 2$

4. $2x^3$

5. 0

6. $\sqrt{2}$

7. $-\dfrac{1}{3}(7x^6 + 2)$

8. $\dfrac{2}{5}x$

9. $-3x^{-4} - 7x^{-8}$

10. $\dfrac{1}{2\sqrt{x}} - \dfrac{1}{x^2}$

11. $24x^{-9} + 1/\sqrt{x}$

12. $-42x^{-7} - \dfrac{5}{2\sqrt{x}}$

13. $12x(3x^2 + 1)$

14. $f(x) = x^{10} + 4x^6 + 4x^2,\ f'(x) = 10x^9 + 24x^5 + 8x$

15. $3ax^2 + 2bx + c$

16. $\dfrac{1}{a}\left(2x + \dfrac{1}{b}\right)$

17. $y' = 10x - 3,\ y'(1) = 7$

18. $y' = \dfrac{1}{2\sqrt{x}} - \dfrac{2}{x^2},\ y'(1) = -3/2$

19. $2t - 1$

20. $\dfrac{1}{3} - \dfrac{1}{3t^2}$

21. $dy/dx = 1 + 2x + 3x^2 + 4x^3 + 5x^4,\ dy/dx|_{x=1} = 15$

22. $\dfrac{dy}{dx} = \dfrac{-3}{x^4} - \dfrac{2}{x^3} - \dfrac{1}{x^2} + 1 + 2x + 3x^2,\ \left.\dfrac{dy}{dx}\right|_{x=1} = 0$

23. $y = (1 - x^2)(1 + x^2)(1 + x^4) = (1 - x^4)(1 + x^4) = 1 - x^8$

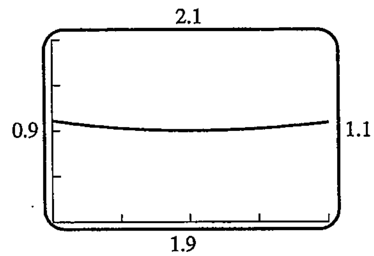
$\dfrac{dy}{dx} = -8x^7$, $dy/dx|_{x=1} = -8$

24. $dy/dx = 24x^{23} + 24x^{11} + 24x^7 + 24x^5$, $dy/dx|_{x=1} = 96$

25. $f'(1) \approx \dfrac{f(1.01) - f(1)}{0.01} = \dfrac{-0.999699 - (-1)}{0.01} = 0.0301$, and by differentiation, $f'(1) = 3(1)^2 - 3 = 0$

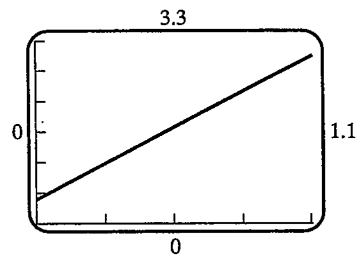
26. $f'(1) \approx \dfrac{f(1.01) - f(1)}{0.01} \approx \dfrac{0.980296 - 1}{0.01} \approx -1.9704$, and by differentiation, $f'(1) = -2/1^3 = -2$

27. From the graph, $f'(1) \approx \dfrac{2.0088 - 2.0111}{1.0981 - 0.9000} \approx -0.0116$,
and by differentiation, $f'(1) = 0$



28. $f(x) = \sqrt{x} + 2x$, $f'(1) \approx \dfrac{3.2488 - 2.7487}{1.1 - 0.9} \approx 2.5006$,

$f'(x) = \dfrac{1}{2\sqrt{x}} + 2$, $f'(1) = \dfrac{5}{2}$



29. $32t$

30. 2π

31. $3\pi r^2$

32. $-2\alpha^{-2} + 1$

33. (a) $\dfrac{dV}{dr} = 4\pi r^2$ (b) $\left.\dfrac{dV}{dr}\right|_{r=5} = 4\pi(5)^2 = 100\pi$

34. $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right] = \dfrac{1}{2 - \lambda_0}\dfrac{d}{d\lambda}(\lambda\lambda_0 + \lambda^6) = \dfrac{1}{2 - \lambda_0}(\lambda_0 + 6\lambda^5) = \dfrac{\lambda_0 + 6\lambda^5}{2 - \lambda_0}$

35. $y - 2 = 5(x + 3)$, $y = 5x + 17$

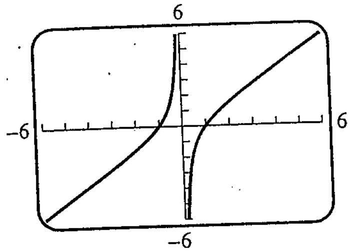
36. $y + 2 = -(x - 2)$, $y = -x$

37. (a) $dy/dx = 21x^2 - 10x + 1$, $d^2y/dx^2 = 42x - 10$
(b) $dy/dx = 24x - 2$, $d^2y/dx^2 = 24$
(c) $dy/dx = -1/x^2$, $d^2y/dx^2 = 2/x^3$
(d) $y = 35x^5 - 16x^3 - 3x$, $dy/dx = 175x^4 - 48x^2 - 3$, $d^2y/dx^2 = 700x^3 - 96x$

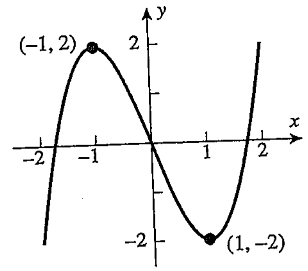
38. (a) $y' = 28x^6 - 15x^2 + 2$, $y'' = 168x^5 - 30x$
(b) $y' = 3$, $y'' = 0$
(c) $y' = \dfrac{2}{5x^2}$, $y'' = -\dfrac{4}{5x^3}$
(d) $y = 2x^4 + 3x^3 - 10x - 15$, $y' = 8x^3 + 9x^2 - 10$, $y'' = 24x^2 + 18x$

58. $dR/dT = 0.04124 - 3.558 \times 10^{-5}T$ which decreases as T increases from 0 to 700. When $T = 0$, $dR/dT = 0.04124\,\Omega/°\text{C}$; when $T = 700$, $dR/dT = 0.01633\,\Omega/°\text{C}$. The resistance is most sensitive to temperature changes at $T = 0°\text{C}$, least sensitive at $T = 700°\text{C}$.

59. $f'(x) = 1 + 1/x^2 > 0$ for all $x \neq 0$



60. $f'(x) = 3x^2 - 3 = 0$ when $x = \pm 1$; increasing for $-\infty < x < -1$ and $1 < x < +\infty$



61. f is continuous at 1 because $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$; also $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-}(2x+1) = 3$ and $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} 3 = 3$ so f is differentiable at 1.

62. f is not continuous at $x = 9$ because $\lim_{x\to 9^-} f(x) = -63$ and $\lim_{x\to 9^+} f(x) = 3$. f cannot be differentiable at $x = 9$, for if it were, then f would also be continuous, which it is not.

63. f is continuous at 1 because $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$, also $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 2x = 2$ and $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} \frac{1}{2\sqrt{x}} = \frac{1}{2}$ so f is not differentiable at 1.

64. f is continuous at 1/2 because $\lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2^+} f(x) = f(1/2)$, also $\lim_{x\to 1/2^-} f'(x) = \lim_{x\to 1/2^-} 3x^2 = 3/4$ and $\lim_{x\to 1/2^+} f'(x) = \lim_{x\to 1/2^+} 3x/2 = 3/4$ so $f'(1/2) = 3/4$, and f is differentiable at $x = 1/2$.

65. (a) $f(x) = 3x - 2$ if $x \geq 2/3$, $f(x) = -3x + 2$ if $x < 2/3$ so f is differentiable everywhere except perhaps at 2/3. f is continuous at 2/3, also $\lim_{x\to 2/3^-} f'(x) = \lim_{x\to 2/3^-}(-3) = -3$ and $\lim_{x\to 2/3^+} f'(x) = \lim_{x\to 2/3^+}(3) = 3$ so f is not differentiable at $x = 2/3$.

(b) $f(x) = x^2 - 4$ if $|x| \geq 2$, $f(x) = -x^2 + 4$ if $|x| < 2$ so f is differentiable everywhere except perhaps at ± 2. f is continuous at -2 and 2, also $\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-}(-2x) = -4$ and $\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+}(2x) = 4$ so f is not differentiable at $x = 2$. Similarly, f is not differentiable at $x = -2$.

66. (a) $f'(x) = -(1)x^{-2}$, $f''(x) = (2\cdot 1)x^{-3}$, $f'''(x) = -(3\cdot 2\cdot 1)x^{-4}$

$$f^{(n)}(x) = (-1)^n \frac{n(n-1)(n-2)\cdots 1}{x^{n+1}}$$

(b) $f'(x) = -2x^{-3}$, $f''(x) = (3\cdot 2)x^{-4}$, $f'''(x) = -(4\cdot 3\cdot 2)x^{-5}$

$$f^{(n)}(x) = (-1)^n \frac{(n+1)(n)(n-1)\cdots 2}{x^{n+2}}$$

47. The y-intercept is -2 so the point $(0,-2)$ is on the graph; $-2 = a(0)^2 + b(0) + c$, $c = -2$. The x-intercept is 1 so the point (1,0) is on the graph; $0 = a + b - 2$. The slope is $dy/dx = 2ax + b$; at $x = 0$ the slope is b so $b = -1$, thus $a = 3$. The function is $y = 3x^2 - x - 2$.

48. Let $P(x_0, y_0)$ be the point where $y = x^2 + k$ is tangent to $y = 2x$. The slope of the curve is $\dfrac{dy}{dx} = 2x$ and the slope of the line is 2 thus at P, $2x_0 = 2$ so $x_0 = 1$. But P is on the line, so $y_0 = 2x_0 = 2$. Because P is also on the curve we get $y_0 = x_0^2 + k$ so $k = y_0 - x_0^2 = 2 - (1)^2 = 1$.

49. The points $(-1,1)$ and $(2,4)$ are on the secant line so its slope is $(4-1)/(2+1) = 1$. The slope of the tangent line to $y = x^2$ is $y' = 2x$ so $2x = 1$, $x = 1/2$.

50. The points $(1,1)$ and $(4,2)$ are on the secant line so its slope is $1/3$. The slope of the tangent line to $y = \sqrt{x}$ is $y' = 1/(2\sqrt{x})$ so $1/(2\sqrt{x}) = 1/3$, $2\sqrt{x} = 3$, $x = 9/4$.

51. $y' = -2x$, so at any point (x_0, y_0) on $y = 1 - x^2$ the tangent line is $y - y_0 = -2x_0(x - x_0)$, or $y = -2x_0x + x_0^2 + 1$. The point $(2,0)$ is to be on the line, so $0 = -4x_0 + x_0^2 + 1$, $x_0^2 - 4x_0 + 1 = 0$. Use the quadratic formula to get $x_0 = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$.

52. Let $P_1(x_1, ax_1^2)$ and $P_2(x_2, ax_2^2)$ be the points of tangency. $y' = 2ax$ so the tangent lines at P_1 and P_2 are $y - ax_1^2 = 2ax_1(x - x_1)$ and $y - ax_2^2 = 2ax_2(x - x_2)$. Solve for x to get $x = \frac{1}{2}(x_1 + x_2)$ which is the x-coordinate of a point on the vertical line halfway between P_1 and P_2.

53. $y' = 3ax^2 + b$; the tangent line at $x = x_0$ is $y - y_0 = (3ax_0^2 + b)(x - x_0)$ where $y_0 = ax_0^3 + bx_0$. Solve with $y = ax^3 + bx$ to get

$$\begin{aligned}
(ax^3 + bx) - (ax_0^3 + bx_0) &= (3ax_0^2 + b)(x - x_0)\\
ax^3 + bx - ax_0^3 - bx_0 &= 3ax_0^2x - 3ax_0^3 + bx - bx_0\\
x^3 - 3x_0^2x + 2x_0^3 &= 0\\
(x - x_0)(x^2 + xx_0 - 2x_0^2) &= 0\\
(x - x_0)^2(x + 2x_0) &= 0, \text{ so } x = -2x_0.
\end{aligned}$$

54. Let (x_0, y_0) be the point of tangency. Note that $y_0 = 1/x_0$. Since $y' = -1/x^2$, the tangent line has the form $y - y_0 = (-1/x_0^2)(x - x_0)$, or $y - \dfrac{1}{x_0} = -\dfrac{1}{x_0^2}x + \dfrac{1}{x_0}$ or $y = -\dfrac{1}{x_0^2}x + \dfrac{2}{x_0}$, with intercepts at $\left(0, \dfrac{2}{x_0}\right)$ and $(2x_0, 0)$. The distance from the y-intercept to the point of tangency is $\sqrt{(0 - x_0)^2 + (y_0 - 2y_0)^2}$, and the distance from the x-intercept to the point of tangency is $\sqrt{(x_0 - 2x_0)^2 + y_0^2}$ so that they are equal (and equal the distance from the point of tangency to the origin).

55. $y' = -\dfrac{1}{x^2}$; the tangent line at $x = x_0$ is $y - y_0 = -\dfrac{1}{x_0^2}(x - x_0)$, or $y = -\dfrac{x}{x_0^2} + \dfrac{2}{x_0}$. The tangent line crosses the x-axis at $2x_0$, the y-axis at $2/x_0$, so that the area of the triangle is $\dfrac{1}{2}(2/x_0)(2x_0) = 2$.

56. $f'(x) = 3ax^2 + 2bx + c$; there is a horizontal tangent where $f'(x) = 0$. Use the quadratic formula on $3ax^2 + 2bx + c = 0$ to get $x = (-b \pm \sqrt{b^2 - 3ac})/(3a)$ which gives two real solutions, one real solution, or none if

(a) $b^2 - 3ac > 0$ (b) $b^2 - 3ac = 0$ (c) $b^2 - 3ac < 0$

57. $F = GmMr^{-2}$, $\dfrac{dF}{dr} = -2GmMr^{-3} = -\dfrac{2GmM}{r^3}$

39. (a) $y' = -5x^{-6} + 5x^4$, $y'' = 30x^{-7} + 20x^3$, $y''' = -210x^{-8} + 60x^2$

(b) $y = x^{-1}$, $y' = -x^{-2}$, $y'' = 2x^{-3}$, $y''' = -6x^{-4}$

(c) $y' = 3ax^2 + b$, $y'' = 6ax$, $y''' = 6a$

40. (a) $dy/dx = 10x - 4$, $d^2y/dx^2 = 10$, $d^3y/dx^3 = 0$

(b) $dy/dx = -6x^{-3} - 4x^{-2} + 1$, $d^2y/dx^2 = 18x^{-4} + 8x^{-3}$, $d^3y/dx^3 = -72x^{-5} - 24x^{-4}$

(c) $dy/dx = 4ax^3 + 2bx$, $d^2y/dx^2 = 12ax^2 + 2b$, $d^3y/dx^3 = 24ax$

41. (a) $f'(x) = 6x$, $f''(x) = 6$, $f'''(x) = 0$, $f'''(2) = 0$

(b) $\dfrac{dy}{dx} = 30x^4 - 8x$, $\dfrac{d^2y}{dx^2} = 120x^3 - 8$, $\left.\dfrac{d^2y}{dx^2}\right|_{x=1} = 112$

(c) $\dfrac{d}{dx}\left[x^{-3}\right] = -3x^{-4}$, $\dfrac{d^2}{dx^2}\left[x^{-3}\right] = 12x^{-5}$, $\dfrac{d^3}{dx^3}\left[x^{-3}\right] = -60x^{-6}$, $\dfrac{d^4}{dx^4}\left[x^{-3}\right] = 360x^{-7}$,

$\left.\dfrac{d^4}{dx^4}\left[x^{-3}\right]\right|_{x=1} = 360$

42. (a) $y' = 16x^3 + 6x^2$, $y'' = 48x^2 + 12x$, $y''' = 96x + 12$, $y'''(0) = 12$

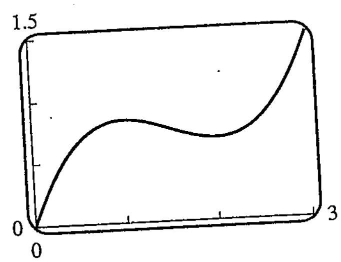
(b) $y = 6x^{-4}$, $\dfrac{dy}{dx} = -24x^{-5}$, $\dfrac{d^2y}{dx^2} = 120x^{-6}$, $\dfrac{d^3y}{dx^3} = -720x^{-7}$, $\dfrac{d^4y}{dx^4} = 5040x^{-8}$,

$\left.\dfrac{d^4y}{dx^4}\right|_{x=1} = 5040$

43. $y' = 3x^2 + 3$, $y'' = 6x$, and $y''' = 6$ so
$y''' + xy'' - 2y' = 6 + x(6x) - 2(3x^2 + 3) = 6 + 6x^2 - 6x^2 - 6 = 0$

44. $y = x^{-1}$, $y' = -x^{-2}$, $y'' = 2x^{-3}$ so
$x^3y'' + x^2y' - xy = x^3(2x^{-3}) + x^2(-x^{-2}) - x(x^{-1}) = 2 - 1 - 1 = 0$

45. The graph has a horizontal tangent at points where $\dfrac{dy}{dx} = 0$, but $\dfrac{dy}{dx} = x^2 - 3x + 2 = (x-1)(x-2) = 0$ if $x = 1, 2$. The corresponding values of y are $5/6$ and $2/3$ so the tangent line is horizontal at $(1, 5/6)$ and $(2, 2/3)$.



46. Find where $f'(x) = 0$; $f'(x) = 1 - 9/x^2 = 0$, $x^2 = 9$, $x = \pm 3$.