REVIEW EXERCISES, CHAPTER 4

1. $\frac{1}{4(6x-5)^{3/4}}(6) = \frac{3}{2(6x-5)^{3/4}}$

2. $\frac{1}{3(x^2+x)^{2/3}}(2x+1) = \frac{2x+1}{3(x^2+x)^{2/3}}$

3. $dy/dx = \frac{3}{2}\left[\frac{x-1}{x+2}\right]^{1/2} \frac{d}{dx}\left[\frac{x-1}{x+2}\right] = \frac{9}{2(x+2)^2}\left[\frac{x-1}{x+2}\right]^{1/2}$

4. $dy/dx = \frac{x^2\frac{4}{3}(3-2x)^{1/3}(-2) - (3-2x)^{4/3}(2x)}{x^4} = \frac{2(3-2x)^{1/3}(2x-9)}{3x^3}$

5. (a) $3x^2 + x\dfrac{dy}{dx} + y - 2 = 0$, $\dfrac{dy}{dx} = \dfrac{2 - y - 3x^2}{x}$

(b) $y = (1 + 2x - x^3)/x = 1/x + 2 - x^2$, $dy/dx = -1/x^2 - 2x$

(c) $\dfrac{dy}{dx} = \dfrac{2 - (1/x + 2 - x^2) - 3x^2}{x} = -1/x^2 - 2x$

6. (a) $xy = x - y$, $x\dfrac{dy}{dx} + y = 1 - \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1-y}{x+1}$

(b) $y(x+1) = x$, $y = \dfrac{x}{x+1}$, $y' = \dfrac{1}{(x+1)^2}$

(c) $\dfrac{dy}{dx} = \dfrac{1-y}{x+1} = \dfrac{1 - \frac{x}{x+1}}{1+x} = \dfrac{1}{x^2+1}$

7. $-\dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{x^2} = 0$ so $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2}$

8. $3x^2 - 3y^2\dfrac{dy}{dx} = 6(x\dfrac{dy}{dx} + y)$, $-(3y^2 + 6x)\dfrac{dy}{dx} = 6y - 3x^2$ so $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{y^2 + 2x}$

9. $\left(x\dfrac{dy}{dx} + y\right)\sec(xy)\tan(xy) = \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{y\sec(xy)\tan(xy)}{1 - x\sec(xy)\tan(xy)}$

10. $2x = \dfrac{(1 + \csc y)(-\csc^2 y)(dy/dx) - (\cot y)(-\csc y \cot y)(dy/dx)}{(1+\csc y)^2}$,

$2x(1 + \csc y)^2 = -\csc y(\csc y + \csc^2 y - \cot^2 y)\dfrac{dy}{dx}$,

but $\csc^2 y - \cot^2 y = 1$, so $\dfrac{dy}{dx} = -\dfrac{2x(1+\csc y)}{\csc y}$

11. $\dfrac{dy}{dx} = \dfrac{3x}{4y}$, $\dfrac{d^2y}{dx^2} = \dfrac{(4y)(3) - (3x)(4dy/dx)}{16y^2} = \dfrac{12y - 12x(3x/(4y))}{16y^2} = \dfrac{12y^2 - 9x^2}{16y^3} = \dfrac{-3(3x^2 - 4y^2)}{16y^3}$,

but $3x^2 - 4y^2 = 7$ so $\dfrac{d^2y}{dx^2} = \dfrac{-3(7)}{16y^3} = -\dfrac{21}{16y^3}$

12. $\dfrac{dy}{dx} = \dfrac{y}{y-x}$,

$\dfrac{d^2y}{dx^2} = \dfrac{(y-x)(dy/dx) - y(dy/dx - 1)}{(y-x)^2} = \dfrac{(y-x)\left(\dfrac{y}{y-x}\right) - y\left(\dfrac{y}{y-x} - 1\right)}{(y-x)^2}$

$= \dfrac{y^2 - 2xy}{(y-x)^3}$ but $y^2 - 2xy = -3$, so $\dfrac{d^2y}{dx^2} = -\dfrac{3}{(y-x)^3}$

13. $\dfrac{dy}{dx} = \tan(\pi y/2) + x(\pi/2)\dfrac{dy}{dx}\sec^2(\pi y/2)$, $\dfrac{dy}{dx} = 1 + (\pi/4)\dfrac{dy}{dx}(2)$, $\dfrac{dy}{dx} = \dfrac{2}{\pi - 2}$

14. Let $P(x_0, y_0)$ be the required point. The slope of the line $4x - 3y + 1 = 0$ is $4/3$ so the slope of the tangent to $y^2 = 2x^3$ at P must be $-3/4$. By implicit differentiation $dy/dx = 3x^2/y$, so at P, $3x_0^2/y_0 = -3/4$, or $y_0 = -4x_0^2$. But $y_0^2 = 2x_0^3$ because P is on the curve $y^2 = 2x^3$. Elimination of y_0 gives $16x_0^4 = 2x_0^3$, $x_0^3(8x_0 - 1) = 0$, so $x_0 = 0$ or $1/8$. From $y_0 = -4x_0^2$ it follows that $y_0 = 0$ when $x_0 = 0$, and $y_0 = -1/16$ when $x_0 = 1/8$. It does not follow, however, that $(0,0)$ is a solution because $dy/dx = 3x^2/y$ (the slope of the curve as determined by implicit differentiation) is valid only if $y \neq 0$. Further analysis shows that the curve is tangent to the x-axis at $(0,0)$, so the point $(1/8, -1/16)$ is the only solution.

15. Substitute $y = mx$ into $x^2 + xy + y^2 = 4$ to get $x^2 + mx^2 + m^2x^2 = 4$, which has distinct solutions $x = \pm 2/\sqrt{m^2 + m + 1}$. They are distinct because $m^2 + m + 1 = (m + 1/2)^2 + 3/4 \geq 3/4$, so $m^2 + m + 1$ is never zero.
Note that the points of intersection occur in pairs (x_0, y_0) and $(-x_0, y_0)$. By implicit differentiation, the slope of the tangent line to the ellipse is given by $dy/dx = -(2x + y)/(x + 2y)$. Since the slope is unchanged if we replace (x, y) with $(-x, -y)$, it follows that the slopes are equal at the two point of intersection.
Finally we must examine the special case $x = 0$ which cannot be written in the form $y = mx$. If $x = 0$ then $y = \pm 2$, and the formula for dy/dx gives $dy/dx = -1/2$, so the slopes are equal.

16. Use implicit differentiation to get $dy/dx = (y - 3x^2)/(3y^2 - x)$, so $dy/dx = 0$ if $y = 3x^2$. Substitute this into $x^3 - xy + y^3 = 0$ to obtain $27x^6 - 2x^3 = 0, x^3 = 2/27$, $x = \sqrt[3]{2}/3$ and hence $y = \sqrt[3]{4}/3$.

17. By implicit differentiation, $3x^2 - y - xy' + 3y^2y' = 0$, so $y' = (3x^2 - y)/(x - 3y^2)$. This derivative exists except when $x = 3y^2$. Substituting this into the original equation $x^3 - xy + y^3 = 0$, one has $27y^6 - 3y^3 + y^3 = 0$, $y^3(27y^3 - 2) = 0$. The unique solution in the first quadrant is $y = 2^{1/3}/3, x = 3y^2 = 2^{2/3}/3$

18. By implicit differentiation, $dy/dx = k/(2y)$ so the slope of the tangent to $y^2 = kx$ at (x_0, y_0) is $k/(2y_0)$ if $y_0 \neq 0$. The tangent line in this case is $y - y_0 = \dfrac{k}{2y_0}(x - x_0)$, or $2y_0y - 2y_0^2 = kx - kx_0$. But $y_0^2 = kx_0$ because (x_0, y_0) is on the curve $y^2 = kx$, so the equation of the tangent line becomes $2y_0y - 2kx_0 = kx - kx_0$ which gives $y_0y = k(x + x_0)/2$. If $y_0 = 0$, then $x_0 = 0$; the graph of $y^2 = kx$ has a vertical tangent at $(0, 0)$ so its equation is $x = 0$, but $y_0y = k(x + x_0)/2$ gives the same result when $x_0 = y_0 = 0$.

19. $y = \ln(x + 1) + 2\ln(x + 2) - 3\ln(x + 3) - 4\ln(x + 4)$, $dy/dx = \dfrac{1}{x+1} + \dfrac{2}{x+2} - \dfrac{3}{x+3} - \dfrac{4}{x+4}$

20. $y = \dfrac{1}{2}\ln x + \dfrac{1}{3}\ln(x + 1) - \ln \sin x + \ln \cos x$, so

$$\frac{dy}{dx} = \frac{1}{2x} + \frac{1}{3(x+1)} - \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{5x+3}{6x(x+1)} - \cot x - \tan x$$

21. $\dfrac{1}{2x}(2) = 1/x$

22. $2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$

23. $\dfrac{1}{3x(\ln x + 1)^{2/3}}$

24. $y = \frac{1}{3}\ln(x + 1)$, $y' = \dfrac{1}{3(x+1)}$

25. $\log_{10} \ln x = \dfrac{\ln \ln x}{\ln 10}$, $y' = \dfrac{1}{(\ln 10)(x \ln x)}$

26. $y = \dfrac{1 + \ln x/\ln 10}{1 - \ln x/\ln 10} = \dfrac{\ln 10 + \ln x}{\ln 10 - \ln x}$, $y' = \dfrac{(\ln 10 - \ln x)/x + (\ln 10 + \ln x)/x}{(\ln 10 - \ln x)^2} = \dfrac{2\ln 10}{x(\ln 10 - \ln x)^2}$

27. $y = \dfrac{3}{2}\ln x + \dfrac{1}{2}\ln(1 + x^4)$, $y' = \dfrac{3}{2x} + \dfrac{2x^3}{(1 + x^4)}$

28. $y = \dfrac{1}{2}\ln x + \ln \cos x - \ln(1 + x^2)$, $y' = \dfrac{1}{2x} - \dfrac{\sin x}{\cos x} - \dfrac{2x}{1 + x^2} = \dfrac{1 - 3x^2}{2x(1 + x^2)} - \tan x$

29. $y = x^2 + 1$ so $y' = 2x$.

30. $y = \ln \dfrac{(1+e^x+e^{2x})}{(1-e^x)(1+e^x+e^{2x})} = -\ln(1-e^x)$, $\dfrac{dy}{dx} = \dfrac{e^x}{1-e^x}$

31. $y' = 2e^{\sqrt{x}} + 2xe^{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = 2e^{\sqrt{x}} + \sqrt{x}e^{\sqrt{x}}$

32. $y' = \dfrac{abe^{-x}}{(1+be^{-x})^2}$

33. $y' = \dfrac{2}{\pi(1+4x^2)}$

34. $y = e^{(\sin^{-1} x)\ln 2}$, $y' = \dfrac{\ln 2}{\sqrt{1-x^2}}2^{\sin^{-1} x}$

35. $\ln y = e^x \ln x$, $\dfrac{y'}{y} = e^x\left(\dfrac{1}{x} + \ln x\right)$, $\dfrac{dy}{dx} = x^{e^x}e^x\left(\dfrac{1}{x}+\ln x\right) = e^x\left[x^{e^x-1} + x^{e^x}\ln x\right]$

36. $\ln y = \dfrac{\ln(1+x)}{x}$, $\dfrac{y'}{y} = \dfrac{x/(1+x) - \ln(1+x)}{x^2} = \dfrac{1}{x(1+x)} - \dfrac{\ln(1+x)}{x^2}$,

$\dfrac{dy}{dx} = \dfrac{1}{x}(1+x)^{(1/x)-1} - \dfrac{(1+x)^{(1/x)}}{x^2}\ln(1+x)$

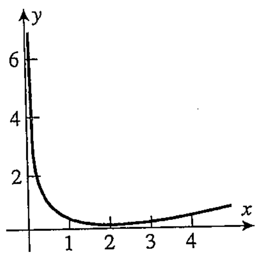
37. $y' = \dfrac{2}{|2x+1|\sqrt{(2x+1)^2-1}}$

38. $y' = \dfrac{1}{2\sqrt{\cos^{-1} x^2}}\dfrac{d}{dx}\cos^{-1} x^2 = -\dfrac{1}{\sqrt{\cos^{-1} x^2}}\dfrac{x}{\sqrt{1-x^4}}$

39. $\ln y = 3\ln x - \dfrac{1}{2}\ln(x^2+1)$, $y'/y = \dfrac{3}{x} - \dfrac{x}{x^2+1}$, $y = \dfrac{3x^2}{\sqrt{x^2+1}} - \dfrac{x^4}{(x^2+1)^{3/2}}$

40. $\ln y = \dfrac{1}{3}(\ln(x^2-1) - \ln(x^2+1))$, $\dfrac{y'}{y} = \dfrac{1}{3}\left(\dfrac{2x}{x^2-1} - \dfrac{2x}{x^2+1}\right) = \dfrac{4x}{3(x^4-1)}$ so $y' = \dfrac{4x}{3(x^4-1)}\sqrt[3]{\dfrac{x^2-1}{x^2+1}}$

41. (b)



(c) $\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{1}{x}$ so $\dfrac{dy}{dx} < 0$ at $x = 1$ and $\dfrac{dy}{dx} > 0$ at $x = e$

(d) The slope is a continuous function which goes from a negative value at $x = 1$ to a positive value at $x = e$; therefore it must take the value zero between, by the Intermediate Value Theorem.

(e) $\dfrac{dy}{dx} = 0$ when $x = 2$

42. $\beta = 10\log I - 10\log I_0$, $\dfrac{d\beta}{dI} = \dfrac{10}{I\ln 10}$

(a) $\left.\dfrac{d\beta}{dI}\right]_{I=10I_0} = \dfrac{1}{I_0\ln 10}$ db/W/m^2

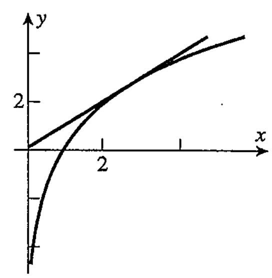
(b) $\left.\dfrac{d\beta}{dI}\right]_{I=100I_0} = \dfrac{1}{10I_0\ln 10}$ db/W/m^2

(c) $\left.\dfrac{d\beta}{dI}\right]_{I=100I_0} = \dfrac{1}{100I_0\ln 10}$ db/W/m^2

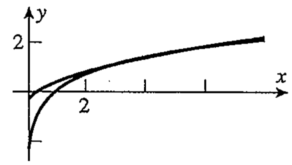
43. Solve $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ given $y = x\ln x$. Then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = (1+\ln x)\dfrac{dx}{dt}$, so $1 + \ln x = 3$, $\ln x = 2$, $x = e^2$.

44. $x = 2, y = 0; y' = -2x/(5 - x^2) = -4$ at $x = 2$, so $y - 0 = -4(x - 2)$ or $y = -4x + 8$

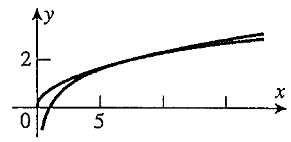
45. Set $y = \log_b x$ and solve $y' = 1$: $y' = \dfrac{1}{x \ln b} = 1$ so $x = \dfrac{1}{\ln b}$. The curves intersect when (x, x) lies on the graph of $y = \log_b x$, so $x = \log_b x$. From Formula (8), Section 1.6, $\log_b x = \dfrac{\ln x}{\ln b}$ from which $\ln x = 1$, $x = e$, $\ln b = 1/e$, $b = e^{1/e} \approx 1.4447$.



46. (a) Find the point of intersection: $f(x) = \sqrt{x} + k = \ln x$. The slopes are equal, so $m_1 = \dfrac{1}{x} = m_2 = \dfrac{1}{2\sqrt{x}}$, $\sqrt{x} = 2$, $x = 4$. Then $\ln 4 = \sqrt{4} + k$, $k = \ln 4 - 2$.



(b) Since the slopes are equal $m_1 = \dfrac{k}{2\sqrt{x}} = m_2 = \dfrac{1}{x}$, so $k\sqrt{x} = 2$. At the point of intersection $k\sqrt{x} = \ln x$, $2 = \ln x$, $x = e^2$, $k = 2/e$.



47. Where f is differentiable and $f' \neq 0$, g must be differentiable; this can be inferred from the graphs. In general, however, g need not be differentiable: consider $f(x) = x^3$, $g(x) = x^{1/3}$.

48. (a) $f'(x) = -3/(x+1)^2$. If $x = f(y) = 3/(y+1)$ then $y = f^{-1}(x) = (3/x) - 1$, so $\dfrac{d}{dx}f^{-1}(x) = -\dfrac{3}{x^2}$; and $\dfrac{1}{f'(f^{-1}(x))} = -\dfrac{(f^{-1}(x)+1)^2}{3} = -\dfrac{(3/x)^2}{3} = -\dfrac{3}{x^2}$.

(b) $f(x) = e^{x/2}$, $f'(x) = \frac{1}{2}e^{x/2}$. If $x = f(y) = e^{y/2}$ then $y = f^{-1}(x) = 2\ln x$, so $\dfrac{d}{dx}f^{-1}(x) = \dfrac{2}{x}$; and $\dfrac{1}{f'(f^{-1}(x))} = 2e^{-f^{-1}(x)/2} = 2e^{-\ln x} = 2x^{-1} = \dfrac{2}{x}$

49. Let $P(x_0, y_0)$ be a point on $y = e^{3x}$ then $y_0 = e^{3x_0}$. $dy/dx = 3e^{3x}$ so $m_{\tan} = 3e^{3x_0}$ at P and an equation of the tangent line at P is $y - y_0 = 3e^{3x_0}(x - x_0)$, $y - e^{3x_0} = 3e^{3x_0}(x - x_0)$. If the line passes through the origin then $(0, 0)$ must satisfy the equation so $-e^{3x_0} = -3x_0e^{3x_0}$ which gives $x_0 = 1/3$ and thus $y_0 = e$. The point is $(1/3, e)$.

50. $\ln y = \ln 5000 + 1.07x$; $\dfrac{dy/dx}{y} = 1.07$, or $\dfrac{dy}{dx} = 1.07y$

51. $\ln y = 2x \ln 3 + 7x \ln 5$; $\dfrac{dy/dx}{y} = 2\ln 3 + 7\ln 5$, or $\dfrac{dy}{dx} = (2\ln 3 + 7\ln 5)y$

52. $$\frac{dk}{dT} = k_0 \exp\left[-\frac{q(T - T_0)}{2T_0T}\right]\left(-\frac{q}{2T^2}\right) = -\frac{qk_0}{2T^2}\exp\left[-\frac{q(T - T_0)}{2T_0T}\right]$$

53. $y' = ae^{ax}\sin bx + be^{ax}\cos bx$ and $y'' = (a^2 - b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx$, so $y'' - 2ay' + (a^2 + b^2)y$
$= (a^2 - b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx - 2a(ae^{ax}\sin bx + be^{ax}\cos bx) + (a^2 + b^2)e^{ax}\sin bx = 0.$