REVIEW EXERCISES, CHAPTER 3

2. (a) $m_{\text{sec}} = \dfrac{f(4) - f(3)}{4 - 3} = \dfrac{(4)^2/2 - (3)^2/2}{1} = \dfrac{7}{2}$

(b) $m_{\tan} = \lim\limits_{x_1 \to 3} \dfrac{f(x_1) - f(3)}{x_1 - 3} = \lim\limits_{x_1 \to 3} \dfrac{x_1^2/2 - 9/2}{x_1 - 3}$

$= \lim\limits_{x_1 \to 3} \dfrac{x_1^2 - 9}{2(x_1 - 3)} = \lim\limits_{x_1 \to 3} \dfrac{(x_1 + 3)(x_1 - 3)}{2(x_1 - 3)} = \lim\limits_{x_1 \to 3} \dfrac{x_1 + 3}{2} = 3$

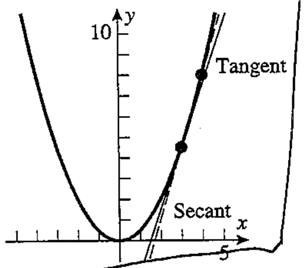
(c) $m_{\tan} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$= \lim\limits_{x_1 \to x_0} \dfrac{x_1^2/2 - x_0^2/2}{x_1 - x_0}$

$= \lim\limits_{x_1 \to x_0} \dfrac{x_1^2 - x_0^2}{2(x_1 - x_0)}$

$= \lim\limits_{x_1 \to x_0} \dfrac{x_1 + x_0}{2} = x_0$

(d)



3. (a) $m_{\tan} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} \dfrac{(x_1^2 + 1) - (x_0^2 + 1)}{x_1 - x_0}$

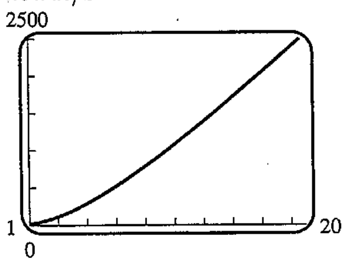
$= \lim\limits_{x_1 \to x_0} \dfrac{x_1^2 - x_0^2}{x_1 - x_0} = \lim\limits_{x_1 \to x_0} (x_1 + x_0) = 2x_0$

(b) $m_{\tan} = 2(2) = 4$

4. To average 60 mi/h one would have to complete the trip in two hours. At 50 mi/h, 100 miles are completed after two hours. Thus time is up, and the speed for the remaining 20 miles would have to be infinite.

5. $v_{\text{inst}} = \lim\limits_{h \to 0} \dfrac{3(h+1)^{2.5} + 580h - 3}{10h} = 58 + \dfrac{1}{10} \dfrac{d}{dx} 3x^{2.5}\Big|_{x=1} = 58 + \dfrac{1}{10}(2.5)(3)(1)^{1.5} = 58.75$ ft/s

6. 164 ft/s



7. (a) $v_{\text{ave}} = \dfrac{[3(3)^2 + 3] - [3(1)^2 + 1]}{3 - 1} = 13$ mi/h

(b) $v_{\text{inst}} = \lim\limits_{t_1 \to 1} \dfrac{(3t_1^2 + t_1) - 4}{t_1 - 1} = \lim\limits_{t_1 \to 1} \dfrac{(3t_1 + 4)(t_1 - 1)}{t_1 - 1} = \lim\limits_{t_1 \to 1} (3t_1 + 4) = 7$ mi/h

9. (a) $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{\sqrt{9-4(x+h)} - \sqrt{9-4x}}{h} = \lim_{h\to 0} \dfrac{9-4(x+h)-(9-4x)}{h(\sqrt{9-4(x+h)}+\sqrt{9-4x})}$

$= \lim_{h\to 0} \dfrac{-4h}{h(\sqrt{9-4(x+h)}+\sqrt{9-4x})} = \dfrac{-4}{2\sqrt{9-4x}} = \dfrac{-2}{\sqrt{9-4x}}$

(b) $\dfrac{dy}{dx} = \lim_{h\to 0} \dfrac{\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}}{h} = \lim_{h\to 0} \dfrac{(x+h)(x+1)-x(x+h+1)}{h(x+h+1)(x+1)}$

$= \lim_{h\to 0} \dfrac{h}{h(x+h+1)(x+1)} = \dfrac{1}{(x+1)^2}$

10. $f(x)$ is continuous and differentiable at any $x \neq 1$, so we consider $x = 1$.

(a) $\lim_{x\to 1^-} (x^2-1) = \lim_{x\to 1^+} k(x-1) = 0 = f(1)$, so any value of k gives continuity at $x = 1$.

(b) $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 2x = 2$, and $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} k = k$, so only if $k = 2$ is $f(x)$ differentiable at $x = 1$.

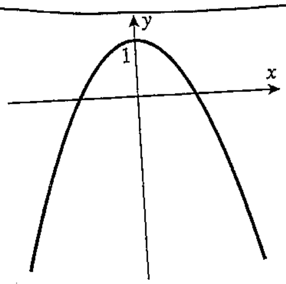
11. (a) $x = -2, -1, 1, 3$

(b) $(-\infty, -2)$, $(-1, 1)$, $(3, +\infty)$

(c) $(-2, -1)$, $(1, 3)$

(d) $g''(x) = f''(x)\sin x + 2f'(x)\cos x - f(x)\sin x$; $g''(0) = 2f'(0)\cos 0 = 2(2)(1) = 4$

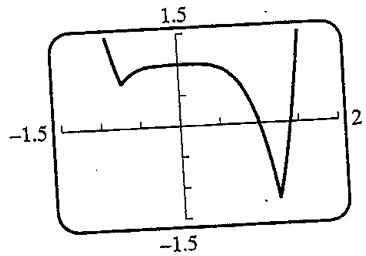
12.



13. (a) The slope of the tangent line $\approx \dfrac{10-2.2}{2050-1950} = 0.078$ billion, or in 2050 the world population was increasing at the rate of about 78 million per year.

(b) $\dfrac{dN/dt}{N} \approx \dfrac{0.078}{6} = 0.013 = 1.3\ \%/\text{year}$

14. When $x^4 - x - 1 > 0$, $f(x) = x^4 - 2x - 1$; when $x^4 - x - 1 < 0$, $f(x) = -x^4 + 1$, and f is differentiable in both cases. The roots of $x^4 - x - 1 = 0$ are $x_1 = -0.724492$, $x_2 = 1.220744$. So $x^4 - x - 1 > 0$ on $(-\infty, x_1)$ and $(x_2, +\infty)$, and $x^4 - x - 1 < 0$ on (x_1, x_2). Then $\lim_{x\to x_1^-} f'(x) = \lim_{x\to x_1^-} (4x^3 - 2) = 4x_1^3 - 2$ and $\lim_{x\to x_1^+} f'(x) = \lim_{x\to x_1^+} -4x^3 = -4x_1^3$ which is not equal to $4x_1^3 - 2$, so f is not differentiable at $x = x_1$; similarly f is not differentiable at $x = x_2$.



15. $f'(x) = 2x\sin x + x^2\cos x$

16. $f'(x) = \dfrac{1 - 2\sqrt{x}\sin 2x}{2\sqrt{x}}$

17. $f'(x) = \dfrac{6x^2 + 8x - 17}{(3x + 2)^2}$

18. $f'(x) = \dfrac{(1 + x^2)\sec^2 x - 2x \tan x}{(1 + x^2)^2}$

19. (a) $\dfrac{dW}{dt} = 200(t - 15)$; at $t = 5$, $\dfrac{dW}{dt} = -2000$; the water is running out at the rate of 2000 gal/min.

(b) $\dfrac{W(5) - W(0)}{5 - 0} = \dfrac{10000 - 22500}{5} = -2500$; the average rate of flow out is 2500 gal/min.

20. (a) $\dfrac{4^3 - 2^3}{4 - 2} = \dfrac{56}{2} = 28$

(b) $(dV/d\ell)|_{\ell=5} = 3\ell^2\big|_{\ell=5} = 3(5)^2 = 75$

21. (a) $f'(x) = 2x, f'(1.8) = 3.6$
(b) $f'(x) = (x^2 - 4x)/(x - 2)^2, f'(3.5) \approx -0.7777778$

22. (a) $f'(x) = 3x^2 - 2x, f'(2.3) = 11.27$
(b) $f'(x) = (1 - x^2)/(x^2 + 1)^2, f'(-0.5) = 0.48$

23. f is continuous at $x = 1$ because it is differentiable there, thus $\lim_{h \to 0} f(1+h) = f(1)$ and so $f(1) = 0$ because $\lim_{h \to 0} \dfrac{f(1+h)}{h}$ exists; $f'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \dfrac{f(1+h)}{h} = 5$.

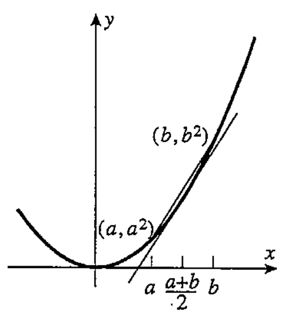
24. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{f(x)f(h) - f(x)}{h} = \lim_{h \to 0} \dfrac{f(x)[f(h) - 1]}{h}$

$= f(x) \lim_{h \to 0} \dfrac{f(h) - f(0)}{h} = f(x)f'(0) = f(x)$

25. The equation of such a line has the form $y = mx$. The points (x_0, y_0) which lie on both the line and the parabola and for which the slopes of both curves are equal satisfy $y_0 = mx_0 = x_0^3 - 9x_0^2 - 16x_0$, so that $m = x_0^2 - 9x_0 - 16$. By differentiating, the slope is also given by $m = 3x_0^2 - 18x_0 - 16$. Equating, we have $x_0^2 - 9x_0 - 16 = 3x_0^2 - 18x_0 - 16$, or $2x_0^2 - 9x_0 = 0$. The root $x_0 = 0$ corresponds to $m = -16, y_0 = 0$ and the root $x_0 = 9/2$ corresponds to $m = -145/4, y_0 = -1305/8$. So the line $y = -16x$ is tangent to the curve at the point $(0, 0)$, and the line $y = -145x/4$ is tangent to the curve at the point $(9/2, -1305/8)$.

26. The slope of the line $x + 4y = 10$ is $m_1 = -1/4$, so we set the negative reciprocal $4 = m_2 = \dfrac{d}{dx}(2x^3 - x^2) = 6x^2 - 2x$ and obtain $6x^2 - 2x - 4 = 0$ with roots $x = \dfrac{1 \pm \sqrt{1 + 24}}{6} = 1, -2/3$.

27. The slope of the tangent line is the derivative $y' = 2x\Big|_{x=\frac{1}{2}(a+b)} = a + b$. The slope of the secant is $\dfrac{a^2 - b^2}{a - b} = a + b$, so they are equal.



28. (a) $f'(1)g(1) + f(1)g'(1) = 3(-2) + 1(-1) = -7$

(b) $\dfrac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \dfrac{-2(3) - 1(-1)}{(-2)^2} = -\dfrac{5}{4}$

(c) $\dfrac{1}{2\sqrt{f(1)}}f'(1) = \dfrac{1}{2\sqrt{1}}(3) = \dfrac{3}{2}$

(d) 0 (because $f(1)g'(1)$ is constant)

29. (a) $8x^7 - \dfrac{3}{2\sqrt{x}} - 15x^{-4}$

(b) $2 \cdot 101(2x+1)^{100}(5x^2 - 7) + 10x(2x+1)^{101}$

(c) $2(x-1)\sqrt{3x+1} + \dfrac{3}{2\sqrt{3x+1}}(x-1)^2$

(d) $3\left(\dfrac{3x+1}{x^2}\right)^2 \dfrac{x^2(3) - (3x+1)(2x)}{x^4} = -\dfrac{3(3x+1)^2(3x+2)}{x^7}$

30. (a) $\cos x - 6\cos^2 x \sin x$

(b) $(1 + \sec x)(2x - \sec^2 x) + (x^2 - \tan x)\sec x \tan x$

(c) $-\csc^2\left(\dfrac{\csc 2x}{x^3+5}\right)\dfrac{-2(x^3+5)\csc 2x \cot 2x - 3x^2 \csc 2x}{(x^3+5)^2}$

(d) $-\dfrac{2 + 3\sin^2 x \cos x}{(2x + \sin^3 x)^2}$

31. Set $f'(x) = 0$: $f'(x) = 6(2)(2x+7)^5(x-2)^5 + 5(2x+7)^6(x-2)^4 = 0$, so $2x + 7 = 0$ or $x - 2 = 0$ or, factoring out $(2x+7)^5(x-2)^4$, $12(x-2) + 5(2x+7) = 0$. This reduces to $x = -7/2$, $x = 2$, or $22x + 11 = 0$, so the tangent line is horizontal at $x = -7/2, 2, -1/2$.

32. Set $f'(x) = 0$: $f'(x) = \dfrac{4(x^2+2x)(x-3)^3 - (2x+2)(x-3)^4}{(x^2+2x)^2}$, and a fraction can equal zero only if its numerator equals zero. So either $x - 3 = 0$ or, after factoring out $(x-3)^3$, $4(x^2+2x) - (2x+2)(x-3) = 0$, $2x^2 + 12x + 6 = 0$, whose roots are (by the quadratic formula) $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 3}}{2} = -3 \pm \sqrt{6}$. So the tangent line is horizontal at $x = 3, -3 \pm \sqrt{6}$.

33. Let $y = mx + b$ be a line tangent to $y = x^2 + 1$ at the point $(x_0, x_0^2 + 1)$ with slope $m = 2x_0$. By inspection if $x_1 = -x_0$ then the same line is also tangent to the curve $y = -x^2 - 1$ at $(-x_1, -y_1)$, since $y_1 = -y_0 = -x_0^2 - 1 = -(x_0^2 + 1) = -x_1^2 - 1$.
Thus the tangent line passes through the points (x_0, y_0) and $(x_1, y_1) = (-x_0, -y_0)$, so its slope $m = \dfrac{y_0 - y_1}{x_0 - x_1} = \dfrac{2y_0}{2x_0} = \dfrac{x_0^2 + 1}{x_0}$. But, from the above, $m = 2x_0$; equate and get $\dfrac{x_0^2+1}{x_0} = 2x_0$, with solution $x_0 = \pm 1$. Thus the only possible such tangent lines are $y = 2x$ and $y = -2x$.

34. (a) Suppose $y = mx + b$ is tangent to $y = x^n + n - 1$ at (x_0, y_0) and to $y = -x^n - n + 1$ at (x_1, y_1). Then $m = nx_0^{n-1} = -nx_1^{n-1}$, and hence $x_1 = -x_0$. Since n is even, $y_1 = -x_1^n - n + 1 = -x_0^n - n + 1 = -(x_0^n + n - 1) = -y_0$. Thus the points (x_0, y_0) and (x_1, y_1) are symmetric with respect to the origin and both lie on the tangent line and thus $b = 0$.

The slope m is given by $m = nx_0^{n-1}$ and by $m = y_0/x_0 = (x_0^n + n - 1)/x_0$, hence $nx_0^n = x_0^n + n - 1$, $(n-1)x_0^n = n - 1$, $x_0^n = 1$. Since n is even, $x_0 = \pm 1$. One easily checks that $y = nx$ is tangent to $y = x^n + n - 1$ at $(1, n)$ and to $y = -x^n - n + 1$ at $(-1, -n)$.

43. **(a)** $\Delta x = 1.5 - 2 = -0.5$; $dy = \dfrac{-1}{(x-1)^2}\Delta x = \dfrac{-1}{(2-1)^2}(-0.5) = 0.5$; and

$\Delta y = \dfrac{1}{(1.5-1)} - \dfrac{1}{(2-1)} = 2 - 1 = 1.$

(b) $\Delta x = 0 - (-\pi/4) = \pi/4$; $dy = \left(\sec^2(-\pi/4)\right)(\pi/4) = \pi/2$; and $\Delta y = \tan 0 - \tan(-\pi/4) = 1$.

(c) $\Delta x = 3 - 0 = 3$; $dy = \dfrac{-x}{\sqrt{25-x^2}} = \dfrac{-0}{\sqrt{25-(0)^2}}(3) = 0$; and

$\Delta y = \sqrt{25-3^2} - \sqrt{25-0^2} = 4 - 5 = -1.$

44. $\cot 46^\circ = \cot \dfrac{46\pi}{180}$; let $x_0 = \dfrac{\pi}{4}$ and $x = \dfrac{46\pi}{180}$. Then

$\cot 46^\circ = \cot x \approx \cot \dfrac{\pi}{4} - \left(\csc^2 \dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right) = 1 - 2\left(\dfrac{46\pi}{180} - \dfrac{\pi}{4}\right) = 0.9651;$

with a calculator, $\cot 46^\circ = 0.9657$.

45. **(a)** $h = 115\tan\phi$, $dh = 115\sec^2\phi\, d\phi$; with $\phi = 51^\circ = \dfrac{51}{180}\pi$ radians and

$d\phi = \pm 0.5^\circ = \pm 0.5\left(\dfrac{\pi}{180}\right)$ radians, $h \pm dh = 115(1.2349) \pm 2.5340 = 142.0135 \pm 2.5340$, so

the height lies between 139.48 m and 144.55 m.

(b) If $|dh| \le 5$ then $|d\phi| \le \dfrac{5}{115}\cos^2 \dfrac{51}{180}\pi \approx 0.017$ radians, or $|d\phi| \le 0.98^\circ$.