Linear Functions of Two Variables

Definition: $z = f(x, y)$ is a **linear** function of two variables if it satisfies:

$$\Delta z = p\Delta x + q\Delta y, \qquad p \text{ and } q \text{ constant}$$
$$z - z_0 = p(x - x_0) + q(y - y_0)$$

The definition of a linear function of two variables is just an extension of the definition of a linear function of one variable. Many other features of linear functions of one and two variables are also similar:

	One Variable	*Two Variables*
Initial Value Form:	$\Delta y = m\Delta x$ $y - y_0 = m \cdot (x - x_0)$ $y(x_0) = y_0$	$\Delta z = p\Delta x + q\Delta y$ $z - z_0 = p \cdot (x - x_0) + q \cdot (y - y_0)$ $z(x_0, y_0) = z_0$
Intercept Form:	$y = mx + b$	$z = px + qy + r$
Constant Rates of Change:	$dy/dx = m$	$\partial z/\partial x = p, \quad \partial z/\partial y = q$
Plots:	graph is a line	surface graph is a plane contour plot with equally spaced levels has (equally spaced) parallel lines, provided not both $p = 0$, $q = 0$. slope of lines in contour plot is $\Delta y/\Delta x = -p/q$

Example

1. Write the linear function satisfying $\Delta z = 4\Delta x + 4\Delta y$, $z(2, 2) = 8$, in intercept form.

Local Linearity

Definitions: $z = f(x, y)$ is *locally linear* at $(x, y) = (x_0, y_0)$ if the surface graph approaches a plane as you zoom in on the point $(x_0, y_0, f(x_0, y_0))$. This plane is called the *tangent plane* to the graph at this point. The equation of this plane *approximates* the function near (x_0, y_0). When comparing changes on the the tangent plane with changes on the function graph, we often write increments on the tangent plane as *differentials*.

Local linearity for a function of two variables is an extension of this concept for a function of one variable:

	One Variable	*Two Variables*
Equation for Tangent:	$dy = mdx$	$dz = pdx + qdy$
(Initial Value Form)	$y - y_0 = m \cdot (x - x_0)$	$z - z_0 = p \cdot (x - x_0) + q \cdot (y - y_0)$
	$m = f'(x_0)$	$p = f_x(x_0, y_0), \quad q = f_y(x_0, y_0)$
	$y_0 = f(x_0)$	$z_0 = f(x_0, y_0)$
Microscope Approximation:	$\Delta y \approx m\Delta x$	$\Delta z \approx p\Delta x + q\Delta y$
	$f(x) - y_0 \approx m \cdot (x - x_0)$	$f(x, y) - z_0 \approx p \cdot (x - x_0) + q \cdot (y - y_0)$
Zooming In:	graph approaches tangent line	surface graph approaches tangent plane
		contour plot approaches contour plot of tangent plane (if tangent plane not horizontal)
		slope of line tangent to level curve at (x_0, y_0):
		$dy/dx = -p/q = -f_x(x_0, y_0)/f_y(x_0, y_0)$

Example

2. $z = f(x, y) = x^2 + y^2$ is locally linear at $(x_0, y_0) = (2, 2)$. Write the equation, in initial value form, of the plane tangent to the surface graph of f at $(x_0, y_0, z_0) = (2, 2, 8)$. Also write the Microscope Approximation for f at about the point $(x_0, y_0) = (2, 2)$. Compare $f(2.01, 1.95)$ with the approximate value obtained using the Microscope Approximation.

Level Sets and Contour Plots

Definition: The **level set** at level c for a function $z = f(x, y)$ of two variables, where c is a constant, is the set

$$\{ (x, y) \in \mathbf{R}^2 \mid f(x, y) = c \}.$$

Notice that the level set is a subset of the domain of f. It is also possible that for some level c the level set has no elements at all (is empty). If the level set is a curve, it is also called a **contour curve**. If it is a line, it is also called a **contour line**.

Definition: A **contour plot** for a function $z = f(x, y)$ of two variables is a collection of level sets for f at certain specified levels c_0, c_1, ... If the level sets are contour curves or lines, the contour plot looks like a topographic map of the surface graph.

3. Suppose $z - z_0 = p \cdot (x - x_0) + q \cdot (y - y_0)$. If $p = 0$ and $q = 0$, what does the surface graph of this linear function look like? What is the level set for the level $c = z_0$? What is the level set for any other value of c?

4. Suppose that $z = px + qy + r$, with $q \neq 0$. Find the equation of the contour line in the (x, y)-plane with $z = c$, a constant. What is its slope? Rewrite your result in terms of partial derivatives of z.

Examples

5. What sort of curves are the contour curves for $z = f(x, y) = x^2 + y^2$?

   Find and plot the contour curves for this function at levels $c = 0$, 1, 4, and 9.

6. In problem 2. above, you determined the equation, in initial value form, of the plane tangent to the surface graph of $f(x, y) = x^2 + y^2$ at $(x_0, y_0, z_0) = (2, 2, 8)$.

   What is the slope of the lines in the contour plot for this tangent plane?

   What is the slope of the line tangent to the level curve passing through $(x_0, y_0) = (2, 2)$ in the contour plot for f itself?

1. $(x_0, y_0) = (2,2)$ and $z_0 = 8$. Therefore $z - 8 = 4(x-2) + 4(y-2)$ which implies $z = 2x + 2y - 8$.

2. $z = f(x,y) = x^2 + y^2$ has partial derivatives $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$. Evaluating at $(x_0, y_0) = (2,2)$, we get $p = 2 \cdot 2 = 4$ and $q = 2 \cdot 2 = 4$. Also, $z_0 = f(x_0, y_0) = f(2,2) = 2^2 + 2^2 = 8$. The equation of the tangent plane to the graph of $f(x,y)$ at the point $(x_0, y_0, z_0) = (2,2,8)$ is thus $z - 8 = 4(x-2) + 4(y-2)$ in initial value form. Thus, the Microscope Approximation for f about $(x_0, y_0) = (2,2)$ is
$$f(x,y) - 8 \approx 4(x-2) + 4(y-2).$$
In particular, $f(2.01, 1.95) \approx 4(2.01-2) + 4(1.95-2) + 8 = 7.84$, while $f(2.01, 1.95) = (2.01)^2 + (1.95)^2 = 7.8426$ if we compute it exactly.

3. $p = 0$ and $q = 0$ implies $z = z_0$. The surface graph is a horizontal plane intercepting the z-axis at the value z_0. The level set for the level $c = z_0$ is this entire plane. There are no points in the level sets for any other values of c.

4. If $z = c$, then $c = px + qy + r$. Solving for y we find $y = -(p/q)x + (c-r)/q$, which is possible since $q \neq 0$. This is the equation of a line in the (x,y)-plane with slope $-p/q$. Note that this slope does not depend on the value of c. In terms of partial derivatives of $z = f(x,y)$, the slope is $-f_x(x,y)/f_y(x,y)$.

5. The contour curves of $z = f(x,y) = x^2 + y^2$ are circles (except that at $c = 0$ it is a single point, and for $c < 0$ there are no curves.)

6. The equation for the tangent plane (from 2.) satisfies
$$dz = 4dx + 4dy.$$
On a horizontal slice producing a contour line, $dz = 0$. Therefore,
$$0 = 4dx + 4dy, \quad \text{so} \quad \frac{dy}{dx} = -\frac{4}{4} = -1$$
is the slope of the contour lines for this tangent plane.
The Microscope Approximation for f about the point $(x_0, y_0) = (2,2)$ satisfies
$$\Delta z \approx 4\Delta x + 4\Delta y.$$
On a horizontal slice producing a contour line, $\Delta z = 0$. Therefore,
$$0 \approx 4\Delta x + 4\Delta y, \quad \text{so} \quad \frac{\Delta y}{\Delta x} \approx -\frac{4}{4} = -1.$$
As we zoom in on the point $(x_0, y_0) = (2,2)$, the error in this approximation approaches zero and the contour curve passing through this point approaches a line with slope $dy/dx = -1$.