Functions of Two Variables

DEFINITION:

Suppose the value of an output variable z is *uniquely* determined once the values of two input variable x and y are given. Then we say that z **is a function of** x **and** y.

As with functions of one variable, to specify the function we need to specify the *domain*, *co-domain*, *name of the function*, and the *rule* assigning a *unique* output value to *each* input pair.

Example

$$f : \mathbf{R}^2 \to \mathbf{R}, \quad z = f(x, y) = 2x^2 + 3xy - 4$$

means that the function *named* f has the set $\mathbf{R}^2$ of *pairs* of real numbers as its domain and the set $\mathbf{R}$ of real numbers as its *co-domain*. The ordered pair of input variables is named (x, y). The output variable is named z. The rule which assigns a value to z given the values of x and y is

$$z = f(x, y) = 2x^2 + 3xy - 4.$$

For this example...

1. If $(x, y) = (1, 2)$, what is the value of z?

2. Can you find a value for x and a value for y which make $z = -4$? Can you find more than one such pair of values? Why doesn't this contradict the claim that f is a *function* (of two variables)?

Surface Graph for a Function of Two Variables

DEFINITION:

Suppose $f : \mathbf{U} \to \mathbf{V}$, $z = f(x, y)$ is a function of two variables.
The **surface graph** of this function is the set

$$\{(x, y, z) \,|\, z = f(x, y) \text{ and } (x, y) \in \mathbf{U}\}.$$

For $f : \mathbf{R}^2 \to \mathbf{R}$, $\quad z = f(x, y) = 2x^2 + 3xy - 4$ ...

3. Explain why both $(x, y, z) = (1, 2, 4)$ and $(x, y, z) = (0, 0, -4)$ are in the surface graph of f.

4. Is $(x, y, z) = (1, 2, 5)$ in the surface graph of f? How do you know?

The surface graph of a function can be partially visualized by drawing part of it as an object in 3-space. This is most easily done with with a computer. A portion of the surface graph of our example function, as plotted by *Derive*, is depicted below. Note the axes labelled x, y, and z.

Vertical Slices for a Function of Two Variables

It's not too easy to draw surface graphs. This is something you can learn to do systematically, but these days it is often easier to use something like *Derive* to produce such a graph.

An alternative approach is to view the graph in *slices*. In this class we will consider *vertical* slices. These slices have the advantage of reducing the problem to graphing functions of one variables.

DEFINITION:

Suppose $f : \mathbf{U} \to \mathbf{V}$, $z = f(x, y)$ is a function of two variables.

The **vertical slice of f along x, holding $y = b$** is the function

$$g(x) = f(x, b), \quad \text{provided } (x, b) \in U.$$

Similarly, the **vertical slice of f along y, holding $x = a$** is the function

$$\phi(y) = f(a, y), \quad \text{provided } (a, y) \in U.$$

For $f : \mathbf{R}^2 \to \mathbf{R}$, $\quad z = f(x, y) = 2x^2 + 3xy - 4$ ...

5. The vertical slice of f along x, holding $y = 3$ is

$$z = f(x, 3) = 2x^2 + 3x \cdot 4 - 4 = 2x^2 - 12x - 4.$$

Find the vertical slice of f along x, holding $y = 0$:

6. The vertical slice of f along y, holding $x = -1$ is

$$z = f(-1, y) = 2(-1)^2 + 3(-1)y - 4 = -3y - 2.$$

Find the vertical slice of f along y, holding $x = 5/2$:

Partial Derivatives for a Function of Two Variables

If you are at a given point (a, b) in the domain of a function of two variable, there are in general *infinitely many different directions* one could move and still be in the domain of the function. You can seek the rate of change of the function as you move in any of these directions.

To keep things simple at first, we will consider only moving in directions parallel to the x or y axes.

7. Suppose you are at the point (a, b) in the domain of a function $f(x, y)$ of two variables. Suppose you move h units parallel to the x axis. What is the x-coordinate of the point you are at then? Has the y-coordinate changed? What is it?

The average rate of change of f between these two points is

$$\frac{f(a+h,b)-f(a,b)}{h}.$$

If we let h get smaller and smaller, the point $(a+h, b)$ approaches the point (a, b). If the difference quotient above approaches a definite limit as $h \to 0$, then we define that as:

DEFINITION:

The partial derivative of f with respect to x, at the point (a, b), is defined as the following limit, if it exists:

$$\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} := f_x(a,b) = \frac{\partial f}{\partial x}(a,b).$$

Similarly, **the partial derivative of f with respect to y, at the point** (a, b), is defined as the following limit, if it exists:

$$\lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k} := f_y(a,b) = \frac{\partial f}{\partial y}(a,b).$$

Note that there are two sorts of notations commonly used for partial derivatives.

8. Compare the definitions of $f_x(a, b)$ and $f_y(a, b)$. Explain why $f_y(a, b)$ is defined by considering points near (a, b) along a line parallel to the y-axis.

Calculating Partial Derivatives

Fortunately, we rarely actually have to evaluate a limit to compute a partial derivative. Your knowledge of derivatives for functions of one variable, along with the idea of vertical slices, is enough! Recall that we defined the vertical slice of f along x, holding $y = b$, as $g(x) = f(x, b)$. Therefore,

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a).$$

TO COMPUTE $f_x(a,b)$, JUST DIFFERENTIATE $f(x,y)$ WITH RESPECT TO x, regarding it as a function of x only (i.e. TREATING y AS IF IT WERE A CONSTANT), then evaluate the result at $x = a$, $y = b$.

Similarly, we defined the vertical slice of f along y, holding $x = a$, as $\phi(y) = f(a, y)$. Therefore,

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = \lim_{k\to 0} \frac{\phi(b+k) - \phi(b)}{k} = \phi'(b).$$

TO COMPUTE $f_y(a,b)$, JUST DIFFERENTIATE $f(x,y)$ WITH RESPECT TO y, regarding it as a function of y only (i.e. TREATING x AS IF IT WERE A CONSTANT), then evaluate the result at $x = a$, $y = b$.

Example: For $f : \mathbf{R}^2 \to \mathbf{R}$, $\quad z = f(x,y) = 2x^2 + 3xy - 4$ ...

9. Find the following partial derivatives:

   $f_x(2,3)$

   $f_y(2,3)$

   $f_x(a,b)$

   $f_y(a,b)$

   $f_x(x,y)$

   $f_y(x,y)$

Answers and Solutions

1. $f(1,2) = 2 \cdot 1^2 + 3 \cdot 1 \cdot 2 - 4 = 4$

2. The equation $-4 = z = f(x,y) = 2x^2 + 3xy - 4$ can be solved by setting $x = 0$ and $y - 0$. It can also be solved by setting $x = 0$ and picking *any* value for y. This does not violate the condition for f to be a function: for each input ordered pair (x,y), there is a unique output value z. It just happens that more than one input ordered pair corresponds to the same output value $z = -4$.

3. The point $(x,y,z) = (1,2,4)$ is on the surface graph of $z = f(x,y) = 2x^2 + 3xy - 4$ because $4 = 2 \cdot 1^2 + 3 \cdot 1 \cdot 2$. Similarly, the point $(x,y,z) = (0,0,-4)$ is on the surface graph of $z = f(x,y) = 2x^2 + 3xy - 4$ because $-4 = 2 \cdot 0^2 + 3 \cdot 0 \cdot 0 - 4$.

4. Since $5 \neq 4 = 2 \cdot 1^2 + 3 \cdot 1 \cdot 2 = f(1,2)$, the point $(x,y,z) = (1,2,5)$ is NOT on the surface graph of f.

5. The vertical slice of $f = 2x^2 + 3xy - 4$ along x, holding $y = 0$, is
$$f(x,0) = 2x^2 + 3x \cdot 0 - 4 = 2x^2 - 4.$$

6. The vertical slice of $f = 2x^2 + 3xy - 4$ along y, holding $x = -1$, is
$$f(-1,y) = 2 \cdot (-1)^2 + 3 \cdot (-1) \cdot y - 4 = -3y - 2.$$

7. In moving h units parallel to the x-axis from the point (a,b), you move to the point $(a+h,b)$. The x-coordinate of this point is $a+h$ while the y-coordinate is b, the same as the y-coordinate of the original point.

8. In defining $f_y(a,b)$, we consider points $(a, b+k)$ which lie along a line parallel to the y axis.

9. For $f : \mathbf{R}^2 \to \mathbf{R}, \quad z = f(x,y) = 2x^2 + 3xy - 4$ ...

   $f_x(x,y) = 4x - 3y,$ $\quad f_y(x,y) = 3x,$

   $f_x(a,b) = 4a - 3b,$ $\quad f_y(a,b) = 3a,$

   $f_x(2,3) = 4 \cdot 2 - 3 \cdot 3 = -1,$ $\quad f_y(2,3) = 3 \cdot 2 = 6.$