Limits at Infinity and Horizontal Asymptotes

We have learned how to apply our ability to differentiate functions in order to find intervals on which functions are increasing and decreasing, as well as classify local maxima and minima, determine concavity and find roots of the function. This is ALMOST enough to sketch a graph by hand for most functions. The last piece we need is the ability to deal with horizontal and vertical asymptotes.

Consider the function $f(x) = \dfrac{4x^2}{x^2+1}$ on the *infinite interval* $(-\infty, +\infty)$.
Let's sketch it below:

x	$-\infty \leftarrow$	-1000	-100	-10	-1	0	1	10	100	1000	$\rightarrow +\infty$
$f(x)$		3.999996	3.9996	3.96	2	0	2	3.96	3.9996	3.999996	

This table suggest that the value of $f(x)$ approaches the value of _____ as x increases without bound (we say $x \to \infty$)
The function $f(x)$ approaches the value of _____ as x *decreases* without bound.
Mathematically, we can write these sentences as:

$$\lim_{x\to\infty} f(x) = 4 \text{ and } \lim_{x\to-\infty} f(x) = 4$$

Thus the function $f(x)$ has horizontal asymptotes at $y = 4$.

Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph $f(x)$ if $\lim\limits_{x\to-\infty}$ or $\lim_{x\to\infty} f(x) = L$

Considering this definition a function can have AT MOST ____ *horizontal* asymptotes.

Some Rules Involving Limits at Infinity

If $\lim\limits_{x\to\infty} f(x) = F$ and $\lim\limits_{x\to-\infty} g(x) = G$ then

1. $\lim\limits_{x\to\infty} [f(x) + g(x)] = F + G$
2. $\lim\limits_{x\to\infty} [f(x) \cdot g(x)] = F \cdot G$
3. If r is a positive real number and c is any real number, $\lim\limits_{x\to\infty} \dfrac{c}{x^r} = 0$

<u>**Exercise**</u>

1. $\lim\limits_{x\to\infty} 7 - \dfrac{1}{\sqrt{x}} =$

2. $\lim\limits_{x\to\infty} \dfrac{x + 300000}{x - 1} =$

3. $\lim\limits_{x\to-\infty} \dfrac{x + 300000}{x - 1} =$

Examples

Evaluate the following limits and compare the results. Notice a pattern?

4. $\lim_{x\to\infty} \frac{3x-5}{4x^2+1}$

5. $\lim_{x\to\infty} \frac{3x^2-5}{4x^2+1}$

6. $\lim_{x\to\infty} \frac{3x^3-5}{4x^2+1}$

Exercise

7. Consider the function $g(x) = \frac{2x-4}{\sqrt{x^2+1}}$.

Q: How many horizontal asymptotes does it have?

A: Evaluate $\lim_{x\to-\infty} \frac{2x-4}{\sqrt{x^2+1}}$ and $\lim_{x\to\infty} \frac{2x-4}{\sqrt{x^2+1}}$

Infinite Limits Involving Trigonometric Functions

Consider the following limits

8. $\lim_{x\to-\infty} \sin(x)$

9. $\lim_{x\to-\infty} \frac{\sin(x)}{x}$

Practice, Practice, Practice

10 (a) $\lim_{x\to\infty} \frac{x}{x^2-1} =$

10 (b) $\lim_{x\to\infty} \frac{5x^2+3}{x-1} =$

10 (c) $\lim_{x\to\infty} \frac{2x}{x-1} + \frac{3x}{x+1} =$

10 (d) $\lim_{x\to\infty} x\cos\left(\frac{1}{x}\right)$

10 (e) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2-x}}$

L'Hôpital's Rule

If the limit on the left has an indeterminate form (i.e. $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$ or $\pm\infty \cdot 0$) then it is equal to the limit on the right (if this limit exists)

$$\lim_{x\to b} \frac{f(x)}{g(x)} = \frac{\lim_{x\to b} f'(x)}{\lim_{x\to b} g'(x)}$$

By using this new rule we can find the limits of a whole bunch of new functions, and we have an easier way to find horizontal asymptotes:

Examples

Take the following limits by first identifying which indeterminate form they take and then apply L'Hopital's Rule.

1. $\lim\limits_{x\to\infty} \dfrac{5+5x}{3x-2}$

2. $\lim\limits_{x\to-\infty} \dfrac{5+5x}{3x-2}$

3. $\lim\limits_{x\to 1} (x-1)^3 \ln(x-1)$

4. $\lim\limits_{x\to 0} \dfrac{\cos(x) - 1 + \frac{1}{2}x^2}{\sin(x) - x}$

Limits of Functions at Undefined x-values

If a function $f(x)$ is defined for all points near an x-value a, but is undefined at a itself, we can ask ourselves what the limit of the function is as x approaches a from either values smaller than a or greater than a or both, i.e. $\lim_{x\to a^-} f(x)$ OR $\lim_{x\to a^+} f(x)$ OR $\lim_{x\to a} f(x)$ is $+\infty$ or $-\infty$
NOTE: Just because the function is undefined at a does not mean the limits will be undefined.

Vertical Asymptotes

A function $f(x)$ has a vertical asymptote at $x = a$ if any of the three limits $\lim\limits_{x\to a^-} f(x)$ OR $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a} f(x)$ is $+\infty$ or $-\infty$

Examples:

For each of the functions below, determine for which x values the function is undefined and thus find out if the function has any **vertical asymptotes** at these points by taking the limit of the function at this point (or points).
(If you have extra time, you should find the **horizontal** asymptotes too.)

5. $f(x) = \dfrac{\sin(x)}{x}$

6. $g(x) = \dfrac{x^2 - 4}{x - 2}$

7. $k(x) = \tan(x)$

8. $m(x) = \dfrac{1}{3x - 2}$

9. $n(x) = \dfrac{5 + 5x}{3x - 2}$

10. $p(x) = \dfrac{(3x + 2)(x - 7)}{(x + 1)(4x + 3)}$

11. $l(x) = e^{\frac{1}{x}}$