Some Elementary Derivative Formulas and Rules

Warm-Up

1. Suppose a function f is differentiable at the point a. Write down the mathematical definition of $f'(a)$.

Derivative of a Constant Function

2. Suppose $f(x) = C$, a constant. Use the definition of the derivative to find $f'(a)$.

The formula you have found works no matter what value a has. That is, given a value of a as *input* you can return the value $f'(a)$ as an output. Viewed from this perspective, f' is a function itself!

3. On the axes to the left, plot the graph of $f(x) = 2$. On the axes to the right, plot the graph of $f'(a)$ for this example.

Derivative of a Linear Function

4. Suppose $f(x) = mx + b$, where m and b are constants. Use the definition of the derivative to find $f'(a)$.

The formula you have found works no matter what value a has. That is, given a value of a as an *input* you can return the value $f'(a)$ as an output. Viewed from this perspective, f' is a function itself!

5. On the axes to the left, plot the graph of $f(x) = 2x + 1$. On the axes to the right, plot the graph of $f'(a)$ for this example.

PROPOSITION: Suppose $f(x) = c \cdot g(x)$, where c is a constant and g is differentiable at a. Then f is differentiable at a and $f'(a) = c \cdot g'(a)$.

Proof:

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} && \text{(definition of the derivative)} \\
&= \lim_{x \to a} \frac{c \cdot g(x) - c \cdot g(a)}{x - a} && \text{(definition of function sums)} \\
&= \lim_{x \to a} c \cdot \frac{g(x) - g(a)}{x - a} && \text{(factoring)} \\
&= c \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} && \text{(property of limits)} \\
&= c \cdot g'(a) && \text{(definition of the derivative).}
\end{aligned}$$

Derivative of the Cosine Function

Suppose $f(x) = \cos(x)$. We will use the definition of the derivative, along with properties of limits and the following three facts, to find $f'(a)$.

Three Facts

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h) \qquad \text{(high school trigonometry)}$$

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0, \qquad \lim_{h\to 0} \frac{\sin(h)}{h} = 1. \qquad \text{(handout, Week 6 Homework)}$$

The Derivation

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} && \text{(definition of the derivative)}\\
&= \lim_{h\to 0} \frac{\cos(a+h)-\cos(a)}{h} && \text{(definition of } f\text{)}\\
&= \lim_{h\to 0} \frac{\cos(a)\cos(h)-\sin(a)\sin(h)-\cos(a)}{h} && \text{(trig identity)}\\
&= \lim_{h\to 0} \frac{\cos(a)\big(\cos(h)-1\big)-\sin(a)\sin(h)}{h} && \text{(factoring)}\\
&= \lim_{h\to 0} \frac{\cos(a)\big(\cos(h)-1\big)}{h} - \lim_{h\to 0} \frac{\sin(a)\sin(h)}{h} && \text{(property of limits)}\\
&= \cos(a)\cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} - \sin(a)\cdot \lim_{h\to 0} \frac{\sin(h)}{h} && \text{(property of limits, since } \cos(a) \text{ and } \sin(a) \text{ are constants)}\\
&= \cos(a)\cdot 0 - \sin(a)\cdot 1 && \text{(known facts)}\\
&= -\sin(a) && \text{(arithmetic).}
\end{aligned}$$

The formula you have found works no matter what value a has. That is, given a value of a as an *input* you can return the value $f'(a)$ as an output. Viewed from this perspective, f' is a function itself!

6. On the axes to the left, plot the graph of $f(x) = \cos(x)$. On the axes to the right, plot the graph of $f'(a)$.

Derivatives of Sums and Differences of Functions

THEOREM: Suppose f and g are differentiable at a. Then $f+g$ is differentiable at a and

$$(f+g)'(a) = f'(a) + g'(a).$$

Proof:

$$\begin{aligned}
(f+g)'(a) &= \lim_{x\to a} \frac{(f+g)(x) - (f+g)(a)}{x-a} && \text{(definition of the derivative)} \\
&= \lim_{x\to a} \frac{\big(f(x)+g(x)\big) - \big(f(a)+g(a)\big)}{x-a} && \text{(definition of function sums)} \\
&= \lim_{x\to a} \frac{f(x) - f(a) + g(x) - g(a)}{x-a} && \text{(algebra)} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} && \text{(property of limits)} \\
&= f'(a) + g'(a) && \text{(definition of the derivative).}
\end{aligned}$$

COROLLARY: If f and g are differentiable at a, then $f-g$ is differentiable at a and

$$(f-g)'(a) = f'(a) - f'(a).$$

Proof:

$$\begin{aligned}
(f-g)'(a) &= \big(f + (-g)\big)'(a) && \text{(subtraction and additive inverses)} \\
&= f'(a) + (-g)'(a) && \text{(the theorem above)} \\
&= f'(a) + (-1\cdot g)'(a) && \text{(property of additive inverses)} \\
&= f'(a) + \big(-1\cdot g'(a)\big) && \text{(the proposition above)} \\
&= f'(a) + \big(-g'(a)\big) && \text{(property of additive inverses)} \\
&= f'(a) - g'(a). && \text{(subtraction and additive inverses)}
\end{aligned}$$

Examples

7. Find the derivatives of the following functions:

   $f(x) = 2\cos(x) + 5x$

   $g(x) = 4x - \cos(x) + \pi$

This handout summarizes information about limits that you have learned or will learn in the second unit of this course.

Definitions

We say that $f(x)$ approaches L as x approaches a *from below*, and write

$$\lim_{x \to a^-} f(x) = L,$$

if and only if it is ALWAYS possible, in principle, to complete a statement of the form

$$\text{“If } \qquad < x < a, \text{ then } 0 \le |f(x) - L| < \text{tolerance,”}$$

no matter how small a positive value we choose for the “tolerance.”

We say that $f(x)$ approaches L as x approaches a *from above*, and write

$$\lim_{x \to a^+} f(x) = L,$$

if and only if it is ALWAYS possible, in principle, to complete a statement of the form

$$\text{“If } a < x < \qquad, \text{ then } 0 \le |f(x) - L| < \text{tolerance,”}$$

no matter how small a positive value we choose for the “tolerance.”

We say that $f(x)$ approaches L as x approaches a, and write

$$\lim_{x \to a} f(x) = L,$$

if and only if the following three statements are true:

$$\lim_{x \to a^-} f(x) \quad \text{exists;}$$

$$\lim_{x \to a^+} f(x) \quad \text{exists;}$$

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Properties

Suppose that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x),$ where c is a constant
2. $\lim_{x\to a} (\, f(x) + g(x)\,) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
4. $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f(x) / \lim_{x\to a} g(x),$ provided $\lim_{x\to a} g(x) \neq 0$
5. (Sandwich Theorem)
   If $f(x) \leq h(x) \leq g(x)$ and $\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x) = L$.

Special Limits

$\lim_{x\to a} C = C$, for any constant C

$\lim_{x\to a} x = a$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a), \quad a > 0$$

$$\lim_{x\to 0} (1 + x\,)^{1/x} = e$$