Limits, Continuity and Differentiability

Warm-Up

Identify the following statements as TRUE or FALSE. If a statement is false, explain why it is false.

1. To say that a function is continuous *means* that its graph doesn't have any sharp corners.

2. If a function is continuous at a point, then it must be differentiable at that point.

3. To evaluate a limit like
$$\lim_{h \to 0} \frac{(h+1)^2 - 1}{h},$$
all you have to do is replace each occurrence of h with 0, then simplify.

Rules for Working With Limits

By working directly with the definition of a limit you have been able to evaluate some limits. However, as expressions become more complicated it can become quite cumbersome to work directly from the definition. The following algebraic rules for working with limits are often very useful. (In a more advanced course you can learn how to derive these from the definition of the limit. The derivations aren't difficult, but they are a little tedious and would distract us from our goal of using limits to investigate continuity and differentiability.)

THEOREM: Suppose that the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

1. $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x)$, where c is a constant
2. $\lim_{x \to a} (\, f(x) + g(x)\,) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
3. $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
4. $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$, provided $\lim_{x \to a} g(x) \neq 0$

PROPOSITION: The following statements are true:

$\lim_{x \to a} C = C$, for any constant C

$\lim_{x \to a} x = a$

Differentiability Implies Continuity

Although not every function continuous at a point is also differentiable there, a function which is differentiable at a point must be continuous at that point. This may be intuitively clear, but we can also prove it formally using the definitions of continuity and differentiability together with some of the properties of limits given in the previous theorem and proposition.

THEOREM: If $f'(a)$ exists, then $\lim_{x \to a} f(x) = f(a)$.

Proof: $f(a)$ is a constant, so $\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} f(x) - \lim_{x \to a} f(a) = \lim_{x \to a} (\, f(x) - f(a)\,)$

$$= \lim_{x \to a} \left((x-a) \frac{f(x) - f(a)}{(x-a)} \right) = \left(\lim_{x \to a} (x-a) \right) \cdot \left(\lim_{x \to a} \frac{f(x) - f(a)}{(x-a)} \right) = 0 \cdot f'(a) = 0.$$

Therefore, $\lim_{x \to a} f(x) - f(a) = 0$, so $\lim_{x \to a} f(x) = f(a)$.

4. Which properties of limits were used in proving this theorem?