Differentiability and Linear Approximation

You have already learned that the derivative of a function f at a point a, if it exists, is the slope of the line tangent to the graph of f at the point $(a, f(a))$. In this class you will learn terms that mathematicians use to refer to various aspects of this concept. You will also learn some properties of the error one incurs in using a tangent line to approximate a function graph.

Differentiability at a Point

A function f is said to be *differentiable at a point* a if its derivative $f'(a)$ exists at that point.

As you saw in the last class, a function f which is differentiable at a has a line tangent to its graph at the point $(a, f(a))$.

Examples

The function $f(x) = x^2 - 1$ is differentiable at $x = 2$. Its derivative there is $f'(2) = 4$, and the line tangent to its graph at the point $(2, f(2)) = (2, 3)$ has the equation $y = 3 + 4 \cdot (x - 2)$.

The function $f(x) = 3x^2$ is differentiable at $x = 1$. Its derivative there is $f'(1) = 6$, and the line tangent to its graph at the point $(1, f(1)) = (1, 3)$ has the equation $y = 3 + 6 \cdot (x - 1)$.

1. Complete the following statement:

   Suppose a function f is differentiable at $x = a$. Its derivative there is $f'(a)$, and the line tangent to its graph at the point $(a, f(a))$ has the equation:

$$y =$$

First-Degree Taylor Polynomial

If a function f is *differentiable at a point* a, then its first degree Taylor polynomial exists and has the formula

$$P_1(x) = f(a) + f'(a)(x - a).$$

Note that the graph of $P_1(x)$ is the line tangent to the graph of f at the point $(a, f(a))$.

Remark: Because the tangent line is a straight line, its slope can be calculated exactly by the "rise over run" formula. To avoid confusion between differences between points on the tangent line and differences between points on the function graph, the differences between point coordinates on the tangent line are denoted dx and dy rather than Δx and Δy. Thus one can write dy/dx for the slope of the tangent line, and as you know

$$\frac{dy}{dx} = f'(a).$$

Thus you will often see the notation dy/dx used for the derivative. (This notation is due to Leibniz, a German mathematician who invented Calculus at about the same time that Newton did.)

The Microscope Approximation

Suppose f is differentiable at a. Let Δx and Δy denote the coordinate differences between the point $(a, f(a))$ and another point (x, y) on the graph of f. That is, $\Delta x = x - a$ and $\Delta y = y - f(a)$. Then since the tangent line approximates the graph of f near $(a, f(a))$,

$$\frac{\Delta y}{\Delta x} \approx f'(a), \quad \text{or equivalently,} \quad \Delta y \approx f'(a)\Delta x.$$

This is referred to as the "Microscope Approximation" because it reflects what one would see by "zooming in" on the graph of f about the point $(a, f(a))$.

Tangent Line Approximation

Suppose f is differentiable at a. Then for values of x near a, the *tangent line approximation* to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a).$$

In this context, the first degree Taylor polynomial $P_1(x) = f(a) + f'(a)(x - a)$ is called the *local linearization* of f about a. Note that the tangent line approximation and the Microscope Approximation are essentially the same thing.

Error in the Tangent Line Approximation

Suppose f is differentiable at a. We will denote the *error in the tangent line approximation* to f at a point x near a by

$$E(x) = f(x) - \big(f(a) + f'(a)(x - a) \big).$$

Example

2. Write an expression for the error $E(x)$ in the tangent line approximation to the graph of $f(x) = x^2 - 1$ about the point $a = 2$.

3. With your graphing calculator, graph $E(x)$. Use this graph to determine $\lim_{x\to 2} E(x)$.

4. With your graphing calculator, graph $E(x)/(x-2)$. Based on this graph, determine $\lim_{x\to 2} E(x)/(x-2)$.

Taylor's Theorem With Remainder

Suppose f is differentiable at a. Then

$$f(x) = f(a) + f'(a)(x-a) + E(x), \quad \text{where } \lim_{x\to a} E(x) = 0 \text{ and } \lim_{x\to a} \frac{E(x)}{x-a} = 0.$$