P R O L O G U E

What Is Mathematics?

What is mathematics? Ask this question of persons chosen at random, and you are likely to receive the answer "Mathematics is the study of number." With a bit of prodding as to what kind of study they mean, you may be able to induce them to come up with the description "the *science* of numbers." But that is about as far as you will get. And with that you will have obtained a description of mathematics that ceased to be accurate some two and a half thousand years ago!

Given such a huge misconception, there is scarcely any wonder that your randomly chosen persons are unlikely to realize that research in mathematics is a thriving, worldwide activity, or to accept a suggestion that mathematics permeates, often to a considerable extent, most walks of present-day life and society.

In fact, the answer to the question "What is mathematics?" has changed several times during the course of history.

Up to 500 B.C. or thereabout, mathematics was indeed the study of number. This was the period of Egyptian and Babylonian mathematics. In those civilizations, mathematics consisted almost solely of arithmetic. It was largely utilitarian, and very much of a 'cookbook' variety. ("Do such and such to a number and you will get the answer.")

From 500 B.C. to 300 A.D. was the era of Greek mathematics. The mathematicians of ancient Greece were primarily concerned with geometry. Indeed, they regarded numbers in a geometric fashion, as measurements of length, and when they discovered that there were lengths to which their numbers did not correspond (the discovery of irrational lengths), their study of number largely came to a halt. For the Greeks, with their emphasis on geometry, mathematics was the study of number *and shape.*

In fact, it was only with the Greeks that mathematics came into being as an area of study, and ceased being a collection of techniques for measuring, counting, and accounting. Greek interest in mathematics was not just utilitarian; they regarded mathematics as an intellectual pursuit having both aesthetic and religious elements. Thales introduced the idea that the precisely stated assertions of mathematics could be logically proved by a formal argument. This innovation marked the birth of the theorem, now the bedrock of mathematics. For the Greeks, this approach culminated in the publication of Euclid's *Elements*, the most widely circulated book of all time after the Bible.

There was no major change in the overall nature of mathematics until the middle of the seventeenth century, when Newton (in England) and Leibniz (in Germany) independently invented the calculus. In essence, the calculus is the study of motion and change. Previous mathematics had been largely restricted to the static issues of counting, measuring, and describing shape. With the introduction of techniques to handle motion and change, mathematicians were able to study the motion of the planets and of falling bodies on earth, the workings of machinery, the flow of liquids, the expansion of gases, physical forces such as magnetism and electricity, flight, the growth of plants and animals, the spread of epidemics, the fluctuation of profits, and so on. After Newton and Leibniz, mathematics became the study of number, shape, *motion, change, and space.*

Most of the initial work involving calculus was directed toward the study of physics; indeed, many of the great mathematicians of the period are also regarded as physicists. But from about the middle of the eighteenth century there was an increasing interest in the mathematics itself, not just its ap-

The first calculus textbook. Isaac Newton wrote this account of his method for the analysis of motion using "fluxions" and infinite series in 1671, but it was not published until 1736, nine years after his death.

plications, as mathematicians sought to understand what lay behind the enormous power that the calculus gave to humankind. By the end of the nineteenth century, mathematics had become the study of number, shape, motion, change, and space, *and of the mathematical tools that are used in this study*.

The explosion of mathematical activity that has taken place in the present century has been dramatic. In the year 1900, all the world's mathematical knowledge would have fitted into about eighty books. Today it would take maybe 100,000 volumes to contain all known mathematics. This extraordinary growth has not only been a furtherance of previous mathematics; many quite new branches of mathematics have sprung up. At the turn of the century, mathematics could reasonably be regarded as consisting of some twelve distinct subjects: arithmetic, geometry, calculus, and so on. Today, between sixty and seventy distinct categories would be a reasonable figure. Some subjects, like algebra or topology, have split into various subfields; others, such as complexity theory or dynamical systems theory, are completely new areas of study.

Given this tremendous growth in mathematical activity, for a while it seemed as though the only simple answer to the question "What is mathematics?" was to say, somewhat fatuously, "It is what mathematicians do for a living." A particular study was classified as mathematics not so much because of *what* was studied but because of *how* it was studied—that is, the methodology used. It was only within the last twenty years or so that a definition of mathematics emerged on which most mathematicians now agree: mathematics is *the science of patterns*. What the mathematician does is examine abstract 'patterns'—numerical patterns, patterns of shape, patterns of motion, patterns of behavior, and so on. Those patterns can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world around us, from the depths of space and time, or from the inner workings of the human mind.

To convey the modern conception of mathematics, this book takes six general themes, covering patterns of counting, patterns of reasoning and communicating, patterns of motion and change, patterns of shape, patterns of symmetry and regularity, and patterns of position (topology).

One aspect of modern mathematics that is obvious to even the casual observer is the use of abstract notations: algebraic expressions, complicated-looking formulas, and geometric diagrams. The mathematician's reliance on abstract notation is a reflection of the abstract nature of the patterns she studies.

Different aspects of reality require different forms of description. For example, the most appropriate way to study the lay of the land or to describe to someone how to find their way around a strange town is to draw a map. Text is far less appropriate. Analogously, line drawings in the form of blueprints are the appropriate way to specify the construction of a building. And musical notation is the most appropriate medium to convey music, apart from, perhaps, actually playing the piece.

In the case of various kinds of abstract, 'formal' patterns and abstract structures, the most appropriate means of description and analysis is mathematics, using mathematical notations, concepts, and procedures. For instance, the symbolic notation of algebra is the most appropriate means of describing and analyzing general behavioral properties of addition and multiplication.

For example, the commutative law for addition could be written in English as:

When two numbers are added, their order is not important.

However, it is usually written in the symbolic form

$$m + n = n + m.$$

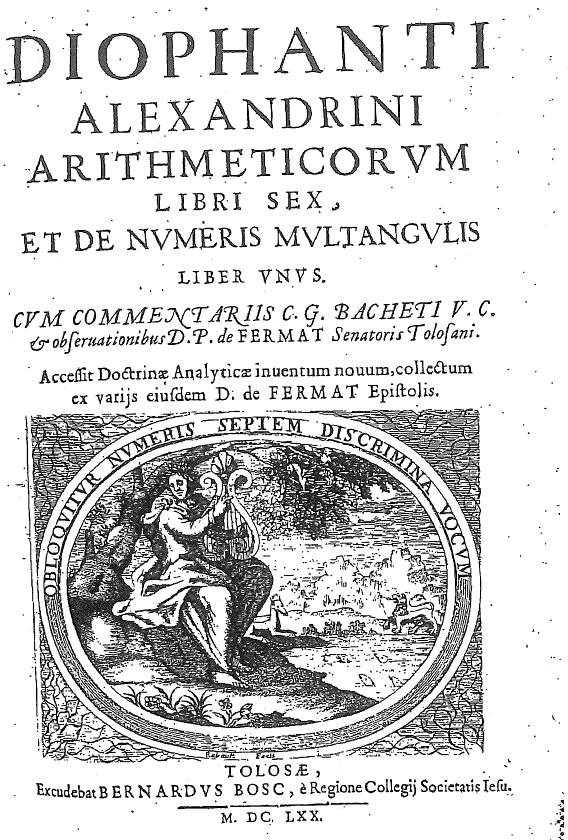
Such is the complexity and the degree of abstraction of the majority of mathematical patterns, that to use anything other than symbolic notation would be prohibitively cumbersome. And so the development of mathematics has involved a steady increase in the use of abstract notations.

But for all that mathematics books tend to be awash with symbols, mathematical notation no more *is* mathematics than musical notation *is* music. A page of sheet music *represents* a piece of music; the music itself is what you get when the notes on the page are sung or performed on a musical instrument. It is in its performance that the music comes alive and becomes part of our experience; the music exists not on the printed page but in our minds. The same is true for mathematics; the symbols on a page are just a representation of the mathematics. When read by a competent performer (in this case, someone trained in mathematics), the symbols on the printed page come alive—the mathematics lives and breathes in the mind of the reader.

Given the strong similarity between mathematics and music, both of which have their own highly abstract notations and are governed by their own structural rules, it is hardly surprising that many (perhaps most) mathematicians also have some musical talent. And yet, until recently, there was a very obvious difference between mathematics and music. Though only someone well trained in music can read a musical score and hear the music in her head, if that same piece of music is performed by a competent musician, anyone with a sense of hearing can appreciate the result. It requires no musical training to experience and enjoy music when it is performed.

But for most of its history, the only way to appreciate mathematics was to learn how to 'sight-read' the symbols. Though the structures and patterns of mathematics reflect the structure of, and resonate in, the human mind every bit as much as do the structures and patterns of music, human beings have developed no mathematical equivalent to a pair of ears. Mathematics can only be 'seen' with the 'eyes of the mind'. It is as if we had no sense of hearing, so that only someone able to sight-read music would be able to appreciate its patterns and its harmonies.

The development of computer and video technologies has to some extent made mathematics accessible to the untrained. In the hands of a skilled user, the computer can be used to 'perform' mathematics, and the result can be displayed on the screen in a visual form for all to see. Though only a relatively small part of mathematics lends itself to such visual 'performance', it is now possible to convey to the layperson at least something of the beauty and the harmony that the mathematician 'sees' and experiences when she does mathematics.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The first systematic use of a recognizably algebraic notation in mathematics seems to have been made by Diophantus, who lived in Alexandria some time around 250 A.D. His treatise *Arithmetic*, of which only six of the original thirteen volumes have been preserved, is generally regarded as the first 'algebra textbook'. In particular, Diophantus used special symbols to denote the unknown in an equation and to denote powers of the unknown, and he employed symbols for subtraction and for equality. The photograph shows the title page of a seventeenth-century Latin translation of Diophantus' classic text.

Like mathematics, music has an abstract notation, used to represent abstract structures.

Without its algebraic symbols, large parts of mathematics simply would not exist. Indeed, the issue is a deep one having to do with human cognitive abilities. The recognition of abstract concepts and the development of an appropriate language are really two sides of the same coin.

The use of a symbol such as a letter, a word, or a picture to denote an abstract entity goes hand in hand with the recognition of that entity *as an entity.* The use of the numeral '7' to denote the number 7 requires that the number 7 be recognized as an entity; the use of the letter '*m*' to denote an arbitrary whole number requires that the *concept* of 'a whole number' be recognized. Having the symbols makes it possible to think about and manipulate the concept.

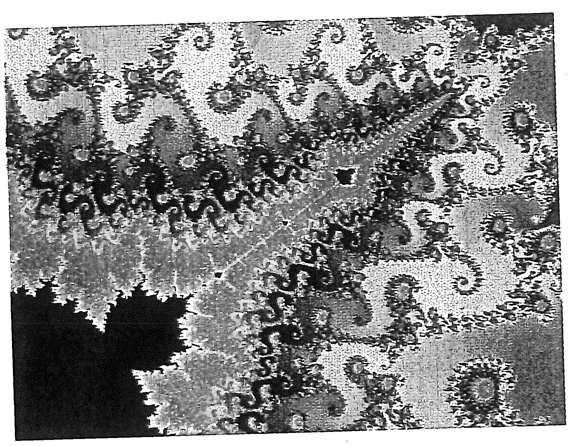
This linguistic aspect of mathematics is often overlooked, especially in our modern culture, with its emphasis on the procedural, computational aspects of mathematics. Indeed, one often hears the complaint that mathematics would be much easier if it weren't for all that abstract notation, which is rather like saying that Shakespeare would be much easier to understand if it were written in simpler language.

Sadly, the level of abstraction in mathematics, and the consequent need for notations that can cope with that abstraction, means that many, perhaps most, parts of mathematics will remain forever hidden from the nonmathematician; and even the more accessible parts—the parts described in books such as this one—may be at best dimly perceived, with much of their inner beauty locked away from view. Still, that is no excuse for those of us who do seem to have been blessed with an ability to appreciate that inner beauty from trying to communicate to others some sense of what it is we experience—some sense of the simplicity, the precision, the purity, and the elegance that give the patterns of mathematics their very considerable aesthetic value.

Mathematical Symphonies

With the aid of modern computer graphics, the mathematician of today can sometimes arrange a 'performance' of mathematics, in much the same way that a musician can perform a piece of music. In this way, the nonmathematician may catch a brief glimpse of the structures that normally live only in the mathematician's mind. Sometimes, the use of computer graphics can be of significant use to the mathematician as well. The study of so-called complex dynamical systems was begun in the 1920s by the French mathematicians Pierre Fatou and Gaston Julia, but it was not until the late 1970s and early 1980s that the rapidly developing technology of computer graphics enabled Benoit Mandelbrot and other mathematicians to see some of the structures Fatou and Julia had been working with. The strikingly beautiful pictures that emerged from this study have since become something of an art form in their own right. In honor of one of the two pioneers of the subject, certain of these structures are now called Julia sets.

The picture is a computer image of part of a fascinating mathematical object discovered by Mandelbrot, now named after him as the Mandelbrot set. The Mandelbrot set is an example of a rich class of objects known as fractals.



In his 1940 book *A Mathematician's Apology*, the accomplished English mathematician G. H. Hardy wrote:

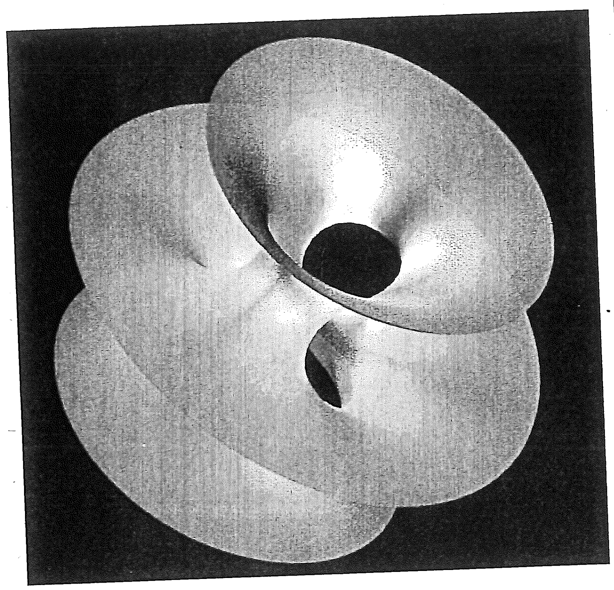
> The mathematician's patterns, like the painter's or the poet's, must be *beautiful*, the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics. . . . It may be very hard to *define* mathematical beauty, but that is just as true of beauty of any kind—we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it.

The beauty to which Hardy was referring is, in many cases, a highly abstract, *inner* beauty, a beauty of abstract form and logical structure, a beauty that can be observed, and appreciated, only by those sufficiently well trained in the discipline. It is a beauty "cold and austere," according to Bertrand Russell, the famous English mathematician and philosopher, who wrote, in his 1918 book *Mysticism and Logic*:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

Mathematics, the science of patterns, is a way of looking at the world, both the physical, biological, and sociological world we inhabit, and the inner world of our minds and thoughts. Mathematics' greatest success has undoubtedly been in the physical domain, where the subject is rightly referred to as both the queen and the servant of the (natural) sciences. Yet, as an entirely human creation, the study of mathematics is ultimately a study of humanity itself. For none of the entities that form the substrate of mathematics exist in the physical world; the numbers, the points, the lines and planes, the surfaces, the

When to See Is to Understand



The mathematician of today can sometimes make use of computer graphics in order to help understand a particular mathematical pattern. The surface shown in this picture was discovered by David Hoffman and William Meeks III in 1983. It is an example of a so-called (non self-intersecting, infinite) minimal surface, the mathematical equivalent of an infinite soap film. Real soap films stretched across a frame always form a surface that occupies the minimal possible area. The mathematician considers abstract analogues that stretch out to infinity. Such surfaces have been studied for over two hundred years, but, until Hoffman and Meeks made their discovery, only three such surfaces were known. Today, as a result of using visualization techniques, mathematicians have discovered many such surfaces.

Much of what is known about minimal surfaces is established by more traditional mathematical techniques, involving lots of algebra and calculus. But, as Hoffman and Meeks showed, the computer graphics can provide the mathematician with the intuition needed to find the right combination of those traditional techniques. A theoretical result by the Brazilian mathematician Celso Costa, in 1983, established the existence of a new infinite minimal surface, but he had no idea what the new surface might look like, or whether it would have the important property of non self-intersection. Using a new computer graphics package developed by James Hoffman (no relation), David Hoffman and Meeks were able to obtain a picture of the strange new surface. Close examination of the picture enabled them to understand the new surface sufficiently well to develop a proof that it did not intersect itself. They were also able to prove that there were in fact infinitely many non self-intersecting, infinite minimal surfaces.

geometric figures, the functions, and so forth are pure abstractions that exist only in humanity's collective mind. The absolute certainty of a mathematical proof and the indefinitely enduring nature of mathematical truth are reflections of the deep and fundamental status of the mathematician's patterns in both the human mind and the physical world.

In an age when the study of the heavens dominated scientific thought, Galileo said, "The great book of nature can be read only by those who know the language in which it was written. And this language is mathematics." Striking a similar note in a much later era, when the study of the inner workings of the atom had occupied the minds of many scientists for a generation, the Cambridge physicist John Polkinghorne wrote, in 1986, "Mathematics is the abstract key which turns the lock of the physical universe."

In today's age, dominated by information, communication, and computation, mathematics is finding new locks to turn. As the science of abstract patterns, there is scarcely any aspect of our lives that is not affected, to a greater or lesser extent, by mathematics; for abstract patterns are the very essence of thought, of communication, of computation, of society, and of life itself.